Exterior derivative

Let M be a smooth manifold and let $\Omega(M) = \oplus_{p\geq 0}\Omega^p(M)$ be the graduated algebra of differential forms on M. Recall that $\wedge^0 T^*M = M \times \mathbb{R}$ so that $\Omega^p(M) = C^\infty(M, \mathbb{R})$. Also $\wedge^1 T^*M = T^*M = \sqcup_{x\in M}(T_xM)^*$.

Theorem. *There is a unique linear map*

$$d : \Omega(M) \to \Omega(M)$$

with

$$d(\Omega^p(M)) \subset \Omega^{p+1}(M),$$

such that

(1) *for $f \in \Omega^0(M)$, $df : M \to T^*M$ is defined by $df_x(v) = T_xf(v)$;*

(2) *for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$ we have*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

(3) $d \circ d = 0$

(4) *Restriction to opens: $(d\omega)|_U = d(\omega|_U)$;*

(5) *for $f : M \to N$ smooth, $f^*(d\omega) = d(f^*\omega)$.*

Proof. *Unicity.*

Lemma. *Let $d : \Omega(M) \to \Omega(M)$ satisfying (1) and (2). If $V \subset M$ is open, $\alpha, \beta \in \Omega(M)$ such that $\alpha|_V = \beta|_V$ then $d\alpha|_V = d\beta|_V$.*
Proof. Take $x \in V$ and $f : M \to \mathbb{R}$ a cut-off function: $\mathit{Supp}\, f \subset V$ and $f \equiv 1$ in a neighborhood of x. Then by the construction $f(\alpha - \beta) = 0$, so that by linearity $d(f(\alpha-\beta)) = 0$. Since d coincides with differentials of functions by (1), and f is constant on V, we have $df_x = 0$. Using (2): $0 = d(f(\alpha-\beta)) = df \wedge(\alpha-\beta) + f \wedge d(\alpha-\beta)$. Evaluating at x and using that $f(x) = 1$ we obtain $0 = 1 \wedge (d\alpha_x - d\beta_x) = d\alpha_x - d\beta_x$ so that $d\alpha_x = d\beta_x$. Since x is arbitrary in V, we obtain the statement in the lemma. □.

Lemma⇒ unicity. We'll express everything in terms of differentials of functions and then use (1). So take $x \in M$ and (U, ϕ) a local chart at x_0, $\phi = (\phi_1, \ldots \phi_n)$. Let $M \to \mathbb{R}$ be a smooth cut-off function with $\mathit{Supp}\, f \subset U$ and $f \equiv 1$ on an open neighborhood V with $x \in V$. For $I = \{i_1, \ldots i_p\}$ with $i_1 < \ldots < i_p$ put

$$d(f\phi)_I = d(f\phi_1) \wedge \ldots \wedge d(f\phi_{i_p}).$$

Write the expression in local coordinates $\omega|_U = \sum_I \omega_I d\phi_I$. Since ω and $\sum_I f\omega_I d(f\phi)_I$ coincide on V, by lemma the differentials are the same. So we compute:

$$d\omega = d(\sum_I f\omega_I d(f\phi)_I) = \sum_I d(f\omega_I) \wedge d(f\phi)_I,$$

where the last equality id obtained using (2) and (3). In this way we expressed $d\omega$ in terms of differentials of functions $f\omega_I$ and $f\phi_{i_j}$, so that we obtain unicity by (1).

Existence

We first consider the case when $M = U$ is an open in $E = \mathbb{R}^n$. Then, for $\omega \in \Omega^p(U) = C^\infty(U, \wedge^p E^*)$ it makes sense to speak about $T_x\omega \in L(\mathbb{R}^n, \wedge^p E^*)$. Then, $\forall v_1, \dots v_{p+1} \in E$ define

$$d\omega_x(v_1, \dots v_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} T_x\omega(v_i)(v_1, \dots, \widehat{v_i}, \dots v_{p+1}).$$

Then, by definition, $d : \Omega^p(U) \to \Omega^{p+1}(U)$ is linear, and coincides with df for f a function. We have the following properties:

1. If $I = \{i_1, \dots i_p\}$ with $i_1 < \dots < i_p$ and if $f \in C^\infty(U, \mathbb{R})$ and $\omega = f dx_I$, then

   $$d\omega = df \wedge dx_I.$$

   In fact, note that dx_I us constant on U, so that $\forall x \in U, v \in E$ we have $T_x(\omega)(v) = df_x(v) dx_I$. Then, by definition of the product, $\forall v_0, \dots v_{p+1} \in E$ we have

   $$d\omega_x(v_1, \dots, v_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} df_x(v_i) dx_I(v_1, \dots \widehat{v_i}, \dots v_{p+1}) = df_x \wedge dx_I(v_1, \dots v_{p+1}).$$

2. $\forall \alpha \in \Omega^p(U), \beta \in \Omega^q(U)$, we have

   $$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

   In fact, by linearity, it's enough to check for $\alpha = f dx_I$ (when I is of cardinal p) and $\beta = g dx_J$ that $\alpha \wedge \beta = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$. Indeed, by the previous property

   $$d(\alpha \wedge \beta) = d(fg) \wedge dx_I \wedge dx_J = g df \wedge dx_I \wedge dx_J + f dg \wedge dx_I \wedge dx_J =$$

   $$= df \wedge dx_I \wedge g dx_J + (-1)^p f dx_I \wedge dg \wedge dx_J = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

3. $d \circ d : \Omega^p(U) \to \Omega^{p+2}(U)$ is a zero map.

   Again, by linearity, it's enough to consider $\omega = f dx_I$, that is similar to the previous property and we leave it as an exercise.

4. Similarly, using linearity and the properties of direct images, one easily sees that, if $V \subset \mathbb{R}^m$ is an open and $\phi : U \to V$ smooth, then $\forall \omega \in \Omega^p(V), d(\phi^* \omega) = \phi^* d\omega$. In particular, $(d\omega)|_V = d(\omega|_V)$.

Now, in the general case, if $\omega \in \Omega(M)$ we cover M by open charts and we define on the chart (U, ϕ)

$$(d\omega)|_U = \phi^*(d(\phi^{-1})^*(\omega|_U)).$$