MATH-GA 2210.001: Homework Local Fields 1

1. Determine all the absolute values for the following fields:

   (a) $k = \mathbb{C}$,

   (b) $k = \mathbb{R}$,

   (c) $k = \mathbb{F}_q$ a finite field with $q = p^r$ elements.

2. Let $| \ |_{p_1}, | \ |_{p_2}, \dots | \ |_{p_k}$ be nontrivial inequivalent absolute values on $\mathbb{Q}$ corresponding to distinct primes p_i, $i = 1, \dots k$, and let $a_1, \dots a_k$ be elements of $\mathbb{Q}$. Let d be the common denominator of a_i. Show that for every $\epsilon > 0$ there is an element $a \in K$ such that $|a - a_i|_{p_i} < \epsilon$ for $i = 1, \dots n$ and $|a|_p < 1/|d|$ for all absolute values corresponding to primes p distinct form p_i, $i = 1, \dots k$.

3. Let k be a field and let $a_1, \dots a_n$ (resp. $b_1, \dots b_n$) be distinct elements of k. Let $K = k(t)$ a purely transcendental extension of k. Show that there exists $x \in K$ such that the functions $x - b_i$ have a simple zero at $t = a_i$ for $i = 1, \dots n$.

4. Let $L = \mathbb{C}(x)[\sqrt{x(x-1)(x+1)}]$ be a degree 2 extension of a purely transcendental extension $\mathbb{C}(x)$ of $\mathbb{C}$, generated by y with $y^2 = x(x-1)(x+1)$. The goal of this exercise is to show that L is not isomorphic to a purely transcendental extension $\mathbb{C}(t)$ of $\mathbb{C}$.

   (a) Let $v : L^* \to \mathbb{Z}$ be a valuation on L. Show that $v(x)$ is even.

   (b) Show that x is not a square in L.

   (c) Let $z \in \mathbb{C}(t)$ be an element, such that for any valuation

   $$v : \mathbb{C}(t)^* \to \mathbb{Z},$$

   one has that $v(z)$ is even. Show that z is a square in $\mathbb{C}(t)$ (Hint: if z is divisible by $t - \alpha$, for $\alpha \in \mathbb{C}$, consider the valuation v given by the order of vanishing at α.)

   (d) Conclude that L is not isomorphic to a purely transcendental extension $\mathbb{C}(t)$ of $\mathbb{C}$.