MATH-GA 2210.001: Homework Analytic Number Theory 2

1. Let $\omega_1, \omega_2 \in \mathbb{C}$, with $\omega_1/\omega_2 \notin \mathbb{R}$. Let f be an entire function in $\mathbb{C}$ satisfying

$$f(z + \omega_1) = af(z) \text{ and } f(z + \omega_2) = bf(z)$$

for some $a, b \in \mathbb{C}$ and all $z \in \mathbb{C}$. Then $f(z) = \alpha \cdot e^{\beta z}$ for some $\alpha, \beta \in \mathbb{C}$.

2. Show that $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{\ln(2\pi)}{2}$.

3. Show that
$$\int_0^\infty \frac{\sin(x)}{x^s} dx = \cos(\pi s/2)\Gamma(1 - s).$$

4. Show that $\log(\Gamma(s))$ has the Taylor expansion around $s = 1$:

$log(\Gamma(s)) = -\gamma(s - 1) + \sum_{n\geq 2} \frac{(-1)^n}{n}\zeta(n)(s - 1)^n$

5. Show that
$$\sum_{n=1}^\infty \frac{\mu(n)}{n} \log(n) = -1.$$