MATH-GA 2210.001: Homework Analytic Number Theory 1

1. Compute the set of Dirichlet characters modulo 8 and modulo 12.

2. Let $S = \{1, 11, 21, 31, \ldots\}$ be the set of positive integers which have the last digit 1 when written in base 10. Prove that S has a natural Dirichlet density, and compute it.

3. Let χ be a Dirichlet character modulo m, $\chi(2) \neq 0$. Show that

$$L(s, \chi) = (1 - 2^{-s}\chi(2))^{-1} \sum_{n=0}^{\infty} \frac{\chi(2n+1)}{(2n+1)^s}.$$

4. Show that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{p} \sum_{m \geq 1} \frac{log\, p}{p^{ms}}$$

for $Re(s) > 1$, where $\zeta(s)$ is the Riemann zeta function.

5. Let γ be the Euler constant:

$$\gamma = \lim_{N \to \infty} 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N} - logN.$$

Verify that

$$\lim_{s \to 1} \zeta(s) - \frac{1}{s-1} = \gamma.$$