**MATH-GA 2150.001: Homework 8**

Let $E$ be an elliptic curve defined over $K = \mathbb{Q}$ by the equation

$$y^2 = x(x-2)(x-10).$$

Denote $\alpha_1 = 0, \alpha_2 = 2, \alpha_3 = 10$.

1. Determine $E(\mathbb{Q})[2]$.
2. Let $S$ the finite set of prime divisors of $\Delta_E = [(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)]^2$. Determine $\mathcal{O}^*_{K,S}/\mathcal{O}^{*2}_{K,S}$.
3. Let $\phi : E(\mathbb{Q})/2E(\mathbb{Q}) \to (\mathcal{O}^*_{K,S}/\mathcal{O}^{*2}_{K,S})^3$ be the embedding :

$$\phi_i(P) = \begin{cases} x_P - \alpha_i & P \neq P_i = (\alpha_i, 0), 0_E \\ (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}), & P = P_i \\ 1 & P = 0_E. \end{cases}$$

   Determine the image by $\phi$ of the 2-torsion points.
4. Show that $\#Im\phi \leq 32$.
5. Show that $(1, -1, -1) \in Im\phi$ and $(1, -3) \in E(\mathbb{Q})$.
6. Show that $(5, 2, 10) \in Im\phi$.
7. Show that if $\gamma_1 \not\equiv 0 (\text{mod } 5)$ and $\gamma_2 \equiv 0 (\text{mod } 5)$, then $(\gamma_1, \gamma_2, \gamma_1\gamma_2) \notin Im\phi$.
8. Show that if $(\gamma_1, \gamma_2, \gamma_1\gamma_2) \notin Im\phi$, then $(5\gamma_1, 2\gamma_2, 10\gamma_1\gamma_2) \notin Im\phi$.
9. Deduce that $\#Im\phi \leq 16$.
10. Show that $(1, 2, 2) \notin Im\phi$.
11. Deduce the value of the rank $E(\mathbb{Q})$.