MATH-GA 2150.001: Homework 6

1. Determine a) $H(\frac{2}{3})$; b) $H(\frac{1+\sqrt{-5}}{2})$.

2. Let K be a number field and let L/K be a finite extension. Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime corresponding to an absolute value v on K. Write $\mathfrak{p}\mathcal{O}_L = \prod\limits_{w|v} \mathfrak{q}_w^{e_w}$ decomposition into prime ideals and set f_w the degree $[\mathcal{O}_L/\mathfrak{q}_w : \mathcal{O}_K/\mathfrak{p}]$.

   (a) Show that $\#\mathcal{O}_L/\mathfrak{p} = (\#\mathcal{O}_K/\mathfrak{p})^{[L:K]}$ (one can use that $\mathcal{O}_K = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$ for $n = [K : \mathbb{Q}]$ and $e_i \in \mathcal{O}_K$, $i = 1 \ldots n$, simialrly for L.)

   (b) Deduce that $[L : K] = \sum e_w f_w$.

   (c) If $x \in \mathcal{O}_K$ show that $ord_{\mathfrak{q}_w} x = e_w ord_{\mathfrak{p}}(x)$. Deduce that $|x|_w = |x|_v^{e_w f_w}$.

   (d) Deduce that for $x_0, \ldots x_n \in K$ one has $\prod_{w|v} max_i |x_i|_w = max_i |x_i|_v^{[L:K]}$.

   (e) Deduce that for $P \in \mathbb{P}^n(K)$ one has $H_L(P) = H_K(P)^{[L:K]}$.