MATH-GA 2150.001: Homework 2

1. Let k be an algebraically closed field. For each of the following plane curves over k write down three open affine charts and determine the intersection with the three coordinate lines ($X = 0, Y = 0$ or $Z = 0$).
   (a) $Y^2Z = X^3 + aXZ^2 + bZ^3$;
   (b) $X^2Y^2 + X^2Z^2 + Y^2Z^2 = 2XYZ(X + Y + Z)$;
   (c) $XZ^3 = (X^2 + Z^2)Y^2$.

2. (a) Let k be a field. Let $P = (0 : 0 : \ldots : 0 : 1) \in \mathbb{P}^n_k$. Show that the set of lines $\mathcal{L}_P$ in $\mathbb{P}^n_k$ passing by P could be identified with a projective space $\mathbb{P}^{n-1}_k$.
   (b) Let $X \subset \mathbb{P}^n_k$ be a quadric : X is a projective variety defined by a homogeneous form $q(x_0, \ldots, x_n)$ of degree 2. Assume that X passes by P and at least one of the derivatives $\partial q/\partial x_i(P)$ is not zero (X is smooth at P). Let T_P be a hyperplane given by the equation $\sum_{i=0}^{n} \partial q/\partial x_i(P)x_i = 0$ (the tangent hyperplane to X at P).
      i. Show that the set of lines in $\mathcal{L}_P$, that are not contained in T_P, is a nonempty open $U_P \subset \mathbb{P}^{n-1}_k$.
      ii. Show that a line $L \in U_{P_0}$ intersects X in exactly two distinct points: P and a second point, that we call P_L.
      iii. Deduce that the projection $U \to X$, $L \mapsto P_L$ is bijective on its image.

3. Let k be an algebraically closed field.
   (a) Show that the set of lines in $\mathbb{P}^2_k$ form a projective space.
   (b) Let $d \geq 2$ be an integer. Consider the set of maps $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$ of degree d. Recall that such a map is given by $(x : y) \mapsto (f_0(x, y) : f_1(x, y) : f_2(x, y))$ where $f_0, f_1, f_2 \in k[x, y]$ are homogeneous polynomials of degree d without a common factor.
      i. Show that the vector of coefficients of f_0, f_1 and f_2 gives a point in a projective space $\mathbb{P}^N_k$, write explicitly N in terms of d.
      ii. Show that the ideal $I = (f_0, f_1, f_2)$ of $k[x, y]$ contains some power of the maximal ideal (x, y).
      iii. For $m \geq 0$ denote $k[x, y]_m$ the set of homogeneous polynomials of degree m in $k[x, y]$. Show that $k[x, y]_m$ is a k-vector space and determine its dimension.
      iv. Consider a map

$$S_m : (k[x, y]_m)^3 \to k[x, y]_{m+d}, (g_0, g_1, g_2) \mapsto \sum_{i=0}^{2} f_i g_i.$$

Show that S_m is a linear map and that if S_m is not surjective then all $(m+d+1)$-minors of some matrix, whose entries are linear combinations of the coefficients of f_0, f_1 and f_2, vanish.

v. Show that for some m the map S_m is surjective.

vi. Deduce that the set maps $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$ of degree d corresponds to a Zariski open in the projective space $\mathbb{P}^N_k$ corresponding to the coefficients of f.