MATH-GA 2150.001: Homework 1

1. Let $J = \langle x^2 + y^2 - 1, y - 1\rangle \subset k[x, y]$.
   (a) Determine $V(J)$.
   (b) Find a function $f \in I(V(J))$ such that $f \notin J$.
   (c) Determine $I(V(J))$.

2. Show that the set $X = \{(x, x) \in \mathbb{R}^2, x \neq 1\}$ is not an affine algebraic variety in $\mathbb{R}^2$.

3. Recall that a topological space X is irreducible if

$$X = F_1 \cup F_2,\ F_1, F_2 \text{ closed in } X \ \Rightarrow F_1 = X \text{ or } F_2 = X.$$

   (a) Show that X is irreducible if and only if for all $U_1, U_2 \subset X$ nonempty open subsets, the intersection $U_1 \cap U_2$ is nonempty.
   (b) Show that X is irreducible if and only if any open subset U of X is dense in X.
   (c) Show that if X is irreducible, then any nonempty open subset U of X is also irreducible.
   (d) Let $X = V(I)$ be an affine variety in k^n. Show that X is irreducible in Zariski topology if and only if $I(X)$ is a prime ideal.
   (e) Let $X = V(I)$ be an affine algebraic variety in k^n. Show that one can write $X = X_1 \cup \ldots \cup X_m$ where X_i are irreducible affine varieties, $X_i \not\subseteq X_j$ if $i \neq j$ and this decomposition is unique up to a permutation of the components.
   (f) Find irreducible components of the following varieties :
      i. $V(y, y^2 - xz) \subset \mathbb{A}^3_k$;
      ii. $V(x(y - x^2 + 1), y(y - x^2 + 1)) \subset \mathbb{A}^2_k$.
      iii. $V(x^2) \subset \mathbb{A}^2_k$

4. Let k be an algebraically closed field. Determine the ideals $I(X)$ of the following algebraic varieties :
   (a) $X = V(x^2 y, (x - 1)(y + 1)^2)$;
   (b) $X = V(y^2 + x^2 y - x^2)$.
   (c) $X = V(z - xy, y^2 + xz - x^2)$.