Algebra I. Homework 5. Due on October 15, 2020.

1. Let R be a ring.
   (a) Give an example of two submodules of an R-module whose union is not a submodule;
   (b) If $\{M_n\}$ is an increasing family of submodules of an R-module M, i.e. $M_n \subset M_p$ for $n \leq p$, show that $\cup M_n$ is a submodule of M.
2. Let R be an integral ring, let K be its field of fractions. Assume $K \neq A$. Prove or disprove: K is a free A-module.
3. Classify all modules of finite type over $\mathbb{Z}/4\mathbb{Z}$.
4. Let α be a complex root of the irreducible polynomial $x^3 - 3x + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in $\mathbb{Q}(\alpha)$ explicitly, in the form $a + b\alpha + c\alpha^2, a, b, c \in \mathbb{Q}$.