Algebra I. Homework 1. Due on September 17, 2020.

1. Prove or disprove: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/4\mathbb{Z}$.

2. Assume that the elements a, b, c of a group G satisfy $abc = e$, where e is the neutral element of G.

   (a) does this imply that $bca = e$?

   (b) does this imply that $bac = e$?

   Give a proof if the answer is positive, give a counterexample if the answer is negative.

3. Determine the number of elements of order 2 in the symmetric group S_4.

4. Classify groups of order 6 by analyzing the following three cases:

   (a) G contains an element of order 6.

   (b) G contains an element of order 3 but not of order 6.

   (c) All elements of G have order 1 or 2.

5. Quaternions. Denote by $\mathbb{H}_8$ the subgroup of $GL_2(\mathbb{C})$, generated by

   $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

   Compute the order of $\mathbb{H}_8$, find all subgroups of $\mathbb{H}_8$, find normal subgroups of H_8 and the corresponding quotient groups.

6. Give an example of a finite group G that satisfies the following properties:

   (a) G is of order 8;

   (b) G is not abelian;

   (c) all the subgroups of G are normal.