3. Three partial orderings of the vertices of a plane graph

Here we follow Schnyder's approach to draw a plane graph on a grid.

Definition 3.1. A barycentric representation of a graph \( G \) is an assignment of triples \( (v_1,v_2,v_3) \) of real numbers to the vertices \( v \in V(G) \) such that

1. \( (v_1,v_2,v_3) = (u_1,u_2,u_3) \Rightarrow v = u \)
2. \( v_1 + v_2 + v_3 = 1 \) \( (\forall v \in V(G)) \)
3. for any \( uv \in E(G) \) and for any \( v \in V(G) \) \( (w \neq u,v) \), there is some \( i \) such that \( w_i > u_i \) and \( w_i > v_i \).

Clearly, any barycentric representation of \( G \) gives rise to a straight-line drawing of \( G \) in the plane \( x + y + z = 1 \) (in \( \mathbb{R}^3 \)).

Proposition 3.2. This straight-line drawing is crossing-free.

Proof. Take two disjoint edges \( uv, wz \in V(G) \). Applying condition (3) to all triples \( uvw, uvz, wzu, wzv \), two of the corresponding \( i \) values coincide. Suppose w.l.o.g. \( w_i > u_i, v_i \) and \( z_i > u_i, v_i \). But then the two edges can be separated by a line. \( \blacksquare \)
In what follows, we consider only maximal plane graphs, that is, triangulations G. Suppose G has a barycentric representation. Then we can label the angles of the interior faces (triangles) of G with the labels 1, 2, 3 as follows:

\[ \forall uvw \text{ gets label } i \iff v_i > u_i, w_i \]

**Homework**: Show that this is a normal labeling.

**Definition 3.3.** A labeling of the vertices of a triangulation G with 1, 2, 3 is called **normal** if

1. the angles of each interior triangle are labeled with 1, 2, and 3, in clockwise order, and
2. at each interior vertex all three labels appear: an interval of 1's followed by an interval of 2's and an interval of 3's, in counterclockwise order.

We will see that every normal labeling can be obtained from a barycentric representation in the above manner.

But first we show that normal labelings of the interior angles is essentially equivalent to normal labelings of the interior edges. More precisely, consider an interior edge uv \in E(G), at which two triangles uvw and vuz meet.
Either at $u$ or at $v$ the two triangles have angles with distinct labels. Suppose w.l.o.g. that the label of $uvw$ is $i$, the label of $uvz$ is $k+j$. Then both angles at $u$ must receive the third label $i$. Now direct the edge $uv$ from $v$ to $u$, and label it with $i$. The directed edges of label $i$ form a graph $T_i$ ($i=1, 2, 3$). These graphs satisfy the conditions of the following definition.

**Definition 3.4.** A partition of the interior edges of $G$ into directed subgraphs $T_1, T_2, T_3$ is called **normal** if

1. the out-degree of each interior vertex is 1 in $T_i$ ($\forall i_i$),
2. the counterclockwise order of the edges around each interior vertex is: outgoing edges in $T_1$, incoming in $T_3$, outgoing in $T_2$, incoming in $T_1$, outgoing in $T_3$, incoming in $T_2$.

**Theorem 3.5.** Any triangulation admits a normal labeling of its interior angles.
Proof. Let \( w \) be a vertex of the exterior triangle in \( G \). By Lemma 1.5, we can find an interior edge \( wz \in E(G) \) that can be contracted without creating any parallel edges other than the images of the two triangles in \( G \) sitting on \( wz \). We contract \( wz \) and show by induction on the number of vertices the stronger statement that there is a labeling in which all interior angles at \( w \) get label 1.

Illustrating the corresponding edge labeling ("normal partition" of edges):

It also follows from this procedure that each \( T_i \) is a directed tree oriented towards a unique exterior vertex, called the root of \( T_i \).

Let \( \overline{T}_i \) denote the directed tree obtained from \( T_i \) by reversing the orientation of every edge. It remains true during the construction that \( T_1 \cup \overline{T}_2 \cup \overline{T}_3 \) has no directed cycle.
For each interior vertex \( v \in V(G) \), follow the path \( P_i(v) \subseteq T_i \) from \( v \) to the root of \( T_i \) \((i=1,2,3)\). These paths cannot meet, apart from the point \( v \), e.g., by the last remark. So they divide the interior of \( G \) into three regions, \( R_1(v), R_2(v), \) and \( R_3(v) \).

Notice that if another interior vertex \( u \in R_i(v) \), then \( R_i(u) \nsubseteq R_i(v) \). (Because of Definition 3.4 (2)!) \( \)

Define \( \overline{V_i} \) \((i=1,2,3)\) as the number of triangles in the region \( R_i(v) \). Clearly, we have \( \overline{V_1}+\overline{V_2}+\overline{V_3}=2n-5 \) for any interior vertex \( v \) (where \( n \) stands for the number of vertices of \( G \)). Extend this definition to the root \( w \) of \( T_i \), by letting \( \overline{W_1}=2n-5 \), \( \overline{W_2}=\overline{W_3}=0 \), and similarly to the roots of \( T_2 \) and \( T_3 \). Now setting \( \overline{v_i} = \frac{\overline{V_i}}{2n-5} \) for every \( v \in V(G) \), \( i=1,2,3 \), we obtain a barycentric representation of \( G \).

\[ \text{Homework: Why?} \]

Let \( p = (1,0), q = (0,1), r = (0,0) \) be regarded as vectors. Then \( \overline{v_1} p + \overline{v_2} q + \overline{v_3} r = (\overline{v_1}, \overline{v_2}) \) is a straight line drawing of \( G \) on a \((2n-5) \times (2n-5)\) grid.