Combinatorics

Midterm Test, October 25, 2006

1. Prove or disprove the following statement: Every graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges is connected (that is, any two vertices of the graph can be connected by a path).

2. A *multigraph* is a graph in which two vertices can be connected by more than one edge. Is it true that the edges of every bipartite multigraph in which every vertex is incident to precisely 10 edges can be colored by 10 colors so that no two edges of the same color share an endpoint?

3. a. How many "words" of length $2n$ can be formed using the letters $a_1, a_1, a_2, a_2, \ldots, a_n, a_n$?

b. How many "words" of length $2n$ can be formed using the above letters such that no two consecutive letters are the same?

4. A graph is called *d-regular* if each of its vertices is incident to precisely d edges.

a. What is the *minimum* number of perfect matchings that a 2-regular graph with n vertices can have?

b. What is the *maximum* number of perfect matchings that a 2-regular graph with n vertices can have?

5. A nested sequence of subsets $A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n$ of the set $\{1, 2, \ldots, n\}$ is called a *complete chain* if $A_0 = \emptyset$, $A_n = \{1, 2, \ldots, n\}$, and in general $|A_i| = i$ for every i $(0 \le i \le n)$.

a. What is the number of complete chains in the set $\{1, 2, \ldots, n\}$?

b. What is the number of complete chains containing a fixed subset $A \subset \{1, 2, \ldots, n\}$?

c. Which subsets A are contained in the largest number of complete chains?

d. Which subsets A are contained in the smallest number of complete chains?

Bonus question: At most how many distinct subsets can be chosen from an n-element set that so that no two of them are disjoint? (For extra credit!)

Please explain all of your answers! Good luck! - J.P.