Unsplittable multiple coverings with balls

A system $\underline{B}=\{B_i : i\in I\}$ of balls in $\mathbb{R}^d$ is said to form a k-<u>fold covering</u> if every point $x\in\mathbb{R}^d$ is covered by at least k elements of $\underline{B}$. A 1-fold covering is simply called a <u>covering</u>. Let $r(B_i)$ and $c(B_i)$ denote the radius and the centre of B_i, respectively. For any two points $x,y\in\mathbb{R}^d$, let $d(x,y)$ stand for their (euclidean) distance.

<u>Theorem 1</u>. For any natural numbers k and $d\geq 3$, there exists a k-fold covering of $\mathbb{R}^d$ with open unit balls $(B_i : i\in I)$, which cannot be decomposed into two coverings. Moreover, we may assume that $\inf_{i\neq j\in I} d(c(B_i),c(B_j)) > 0$.

<u>Theorem 2</u>. For any natural numbers k and $d\geq 3$, there exists a constant $c_d>0$ such that every k-fold covering of $\mathbb{R}^d$ with unit balls $\underline{B}=\{B_i : i\in I\}$, having the property that no point of $\mathbb{R}^d$ is covered by more than $c_d 2^{k/d}$ members of $\underline{B}$, can be decomposed into two coverings.

2. Proof of Theorem 1

The following construction will be basic for our purposes.

Theorem 2.1. For every natural number k there exist a finite point system P_k and a finite system of (not necessarily equal) closed discs $\underline{D}_k$ in the plane with the property that for any 2-colouring of the elements of P_k we can find a disc $D \in \underline{D}_k$ such that

(i) $|D \cap P_k| \geq k$,

(ii) $D \cap P_k$ is monochromatic.

Proof. Given any disc D in the plane, let c(D) and r(D) denote the center and the radius of D, respectively.

If k=1 then the statement is trivial. Let $k \geq 2$ be fixed. We are going to construct $\underline{D}_i = \underline{D}_i' \cup \underline{D}_i''$ and P_i ($i \leq k$) by recursion.

Let D^* be a disc of radius $\delta_2 = 1/10$, and let $\underline{D}_2' = (D^1, D^2, \ldots, D^k)$ be a system of unit discs such that $d(c(D^s), c(D^t))$ (i.e., the distance between the centers of D^s and D^t) is at most δ_2 and each D^t has exactly one point (say, p^t) in common with D^*. Set $\underline{D}_2'' = (D^*)$, $\underline{D}_2 = \underline{D}_2' \cup \underline{D}_2''$ and $P_2 = \{c(D), p^1, p^2, \ldots, p^k\}$.

Assume now that $\underline{D}_i = \underline{D}_i' \cup \underline{D}_i''$ and P_i have already been defined for some $2 \leq i < k$, and

(1) every $D \in \underline{D}_i'$ has a boundary point p(D) not contained in any other element of

An easy continuity argument shows that there exists a sufficiently small positive number

$$\delta_{i+1} \leq \frac{1}{10} \min\{\delta_i, \min_{D \in \underline{D}_i'} d(p(D), \bigcup_{D \neq \tilde{D} \in \underline{D}_i} \tilde{D})\} \tag{2}$$

such that for any $D \in \underline{D}_i'$ one can find k distinct unit discs $D^1, D^2, \ldots, D^k$ and a disc D^* of radius δ_{i+1} with $c(D^*) = p(D)$ satisfying

(3) $$d(c(D^t),c(D)) \leq \delta_{i+1} ,$$

(4) $$D^t \cap P_i = (\mathrm{int}D^t) \cap P_i = D \cap P_i ,$$

(5) D^t and D^* have exactly one point (say, $p^t(D)$) in common,

for all $1 \leq t \leq k$. Set

(6) $$\underline{D}'_{i+1}=(D^1,D^2,\dots,D^k : D\in\underline{D}'_i),\ \underline{D}''_{i+1}=\underline{D}''_i \cup (D^* : D\in\underline{D}'_i),\ \underline{D}_{i+1}=\underline{D}'_{i+1}\cup\underline{D}''_{i+1},$$
$$P_{i+1}=P_i \cup (p^1(D),p^2(D),\dots,p^k(D) : D\in\underline{D}'_i).$$

Let D^t be any element of $\underline{D}'_{i+1}$ ($D\in\underline{D}'_i$, $1\leq t\leq k$) and let E be any element of $\underline{D}_{i+1}$ distinct from D^t and D^*. If $E\in\underline{D}''_i$ then, by (2),

$$d(p^t(D),E)\geq d(p(D),E)-d(p(D),p^t(D))\geq 10\delta_{i+1}-\delta_{i+1}>0.$$

Similarly, if $E=\tilde{D}^s$ or $E=\tilde{D}^*$ for some $\tilde{D}\neq D$ ($\tilde{D}\in\underline{D}'_i$, $1\leq s\leq k$), then

$$d(p^t(D),E)\geq d(p(D),\tilde{D})-d(p(D),p^t(D))-d(c(E),c(\tilde{D}))\geq 8\delta_{i+1}>0$$

and

$$d(p^t(D),E)\geq d(p(D),\tilde{D})-d(p(D),p^t(D))-r(E)\geq 8\delta_{i+1}>0,$$

respectively. If $E=D^s$ for some $s\neq t$, then $d(p^t(D),E)$ is clearly positive. This shows that $p^t(D)$ is not covered by any element of $\underline{D}_{i+1}$ different from D^t and D^*, so we can find a boundary point of D^t sufficiently close to $p^t(D)$ which does not belong to any other element of $\underline{D}_{i+1}$. Thus, (1) remains valid for $i+1$, and the algorithm can be repeated.

Let D be any element of $\underline{D}'_i$. By (2)-(6) it follows that

(7) $$D^t \cap P_{i+1} = (D\cap P_i)\cup\{p^t(D)\} \qquad (1\leq t\leq k),$$
$$D^* \cap P_{i+1} = D^* \cap P_k = \{p^1(D),p^2(D),\dots,p^k(D)\}.$$

We will prove by induction on i that $\underline{D}_i$ and P_i ($2\leq i\leq k$) meet the requirements of the theorem. For $i=2$ this is obviously true. Let $2\leq i<k$, and let $f: P_{i+1}\to\{B,W\}$ be any colouring of the elements of P_{i+1} with 2 colours (Black and White). Applying the induction hypothesis to $f|_{P_i}$ (the restriction of f to P_i), we obtain that there exists a $D\in\underline{D}_i$ such

that $D\cap P_i$ is monochromatic, i.e., $f(x)=f(y)$ for any $x,y\in D\cap P_i$.

Assume first that $D\in\underline{D}'_i$ and all elements of $D\cap P_i$ are, say, black. If $f(p^t(D))=B$ for some $1\le t\le k$, then, by (7), $D^t\cap P_{i+1}=(D\cap P_i)\cup(p^t(D))$ is monochromatic. If $f(p^t(D))=W$ for all $1\le t\le k$, then $D^*\cap P_{i+1}$ is coloured completely white. Since $D^t,D^*\in\underline{D}_{i+1}$, $|D^t|=i+1$ and $|D^*|=k\ge i+1$, in both cases there exists a disc in $\underline{D}_{i+1}$ containing at least $i+1$ points of P_{i+1}, which are coloured the same.

Suppose next that $D\in\underline{D}''_i$. Then, by (7), $D\cap P_{i+1}=D\cap P_i$ and $|D\cap P_{i+1}|=k\ge i+1$. Thus, in this case $D\in\underline{D}''_i\subseteq\underline{D}_{i+1}$ satisfies conditions (i) and (ii) of the theorem with $i+1$ instead of k. This completes the induction, and hence the proof. □

We will make use of the following special feature of our construction.

Lemma 2.2. Let $\underline{D}_k$ and P_k denote the same as above. For any $D_1\in\underline{D}''_k$, set

$$X(D_1)=(\mathrm{bd}\ D_1)\cap\Big(\bigcup_{D_1\neq E\in\underline{D}_k}E\Big).$$

Then $X(D_1)$ can be covered by an angular region of size at most $2\pi/3$, whose apex is at the centre of D_1.

Proof. Let $i+1$ be the smallest integer such that $D_1\in\underline{D}''_{i+1}$. Then $D_1=D^*$ for some $D\in\underline{D}'_i$ $(1\le i<k)$, $r(D^*)=\delta_{i+1}$ and, by (2), D^* is obviously disjoint from all other elements of $\underline{D}''_{i+1}$. Similarly, no element of $\underline{D}''_j$ $(j>i+1)$ intersects D^*.

By the first part of (6), for any $E\in\underline{D}'_k$ there exist a sequence of unit discs $(E_i,E_{i+1},\dots,E_k=E)$ and a sequence of integers $(t_i,t_{i+1},\dots,t_{k-1})$ such that $E_j\in\underline{D}'_j$, $1\le t_j\le k$ and

$$E_{j+1}=(E_j)^{t_j}\qquad(i\le j<k).\tag{8}$$

If $E_i\neq D$ then, by (2), (3) and (8),

$$d(c(D^*),E)=d(p(D),E_k)\ge d(p(D),E_i)-\sum_{i\le j<k}d(c(E_j),c(E_{j+1}))$$

$$\ge 106\delta_{i+1}-\sum_{i\le j<k}\delta_{j+1}>6\delta_{i+1}=r(D^*),$$

so E is disjoint from D^*.

Assume next that $E_i=D$, and let x denote the intersection point of bd D^* and the segment $[c(D),c(D^*)]=[c(D),p(D)]$. By (3),

$$d(x,p^t(D)) \le 2\delta_{i+1}\frac{\delta_{i+1}}{\delta_{i+1}+1} \qquad (1\le t\le k)$$

On the other hand,

$$d(c(E_{i+1}),c(E)) \le \sum_{i<j<k} d(c(E_j),c(E_{j+1})) \le \sum_{i<j<k} \delta_{j+1} < \delta_{i+1}/8.$$

Easy trigonometric calculations show that for any $y\in(\text{bd } D^*)\cap E$

$$d(x,y) \le d(x,p^{t_i}(D)) + d(p^{t_i}(D),y) \le \frac{\delta_{i+1}}{5} + (2d(c(E_{i+1}),c(E))\delta_{i+1})^{1/2}$$
$$\le \frac{3}{4}\delta_{i+1},$$

and the result follows. □

In what follows, we will turn the above planar construction into a 3-dimensional arrangement of balls.

We will think of $\mathbb{R}^2$ as of a horizontal plane in $\mathbb{R}^3$. The terms 'above' and 'below' will be used in this sense.

Lemma 2.3. Let $\underline{D}_k$ and $P_k\subset\mathbb{R}^2$ be the same as above, $R>1$. For every $D\in\underline{D}_k$, let $B(D)\subset\mathbb{R}^3$ be a closed ball of radius R, whose centre is above $\mathbb{R}^2$ and $B(D)\cap\mathbb{R}^2=D$. If B is any closed ball of radius R such that

$$B \subseteq \bigcup_{D\in\underline{D}_k} B(D),$$

then $B = B(D)$ for some $D\in\underline{D}_k$.

Proof. Let B be a closed ball of radius R, which is completely covered by B(D)'s, and set $D_o=B\cap\mathbb{R}^2$. Obviously, the centre of B must be above $\mathbb{R}^2$, and $r(D_o) \le \max_{D\in\underline{D}_k} r(D) = 1$, otherwise the south pole of B cannot be covered by any B(D).

Pick a disc $D_1\in\underline{D}_k$ such that $B(D_1)$ covers the north pole of B. Then $r(D_o)\ge r(D_1)$ with equality if and only if $D_o=D_1$ and hence $B=B(D_1)$.

Thus, we may assume that $r(D_1)<r(D_0)\leq 1$, which implies that $D_1\in\underline{D}_k''$. From the fact that $B(D_1)$ covers the north pole of B it follows that B covers the south pole of $B(D_1)$. Consequently, $D_0=B\cap\mathbb{R}^2$ contains $c(D_1)$ in its interior. This implies that D_0 covers a circular arc of bd D_1 whose angle is larger than $2\pi/3$, which contradicts Lemma 2.2, because

$$(\text{bd } D_1)\cap D_0 \subseteq (\text{bd } D_1)\cap\Big(\bigcup_{D_1\neq E\in\underline{D}_k} E\Big) = X(D_1). \qquad \square$$

Remark 2.4. Enlarging some circles in $\underline{D}_k$ a little bit, we can obviously attain that $D\cap P_k=(\text{int } D)\cap P_k$ and all other properties remain valid. It is also clear that in this case Lemma 2.3 can be stated in the following 'quantitative' form. There exists a small constant $\varepsilon_k>0$ with the property that, for any ball B of radius R, we can either find a disc $D\in\underline{D}_k$ such that

$$B\cap P_k = D\cap P_k = (\text{int } D)\cap P_k ,$$

or $B\setminus\big(\bigcup_{D\in\underline{D}_k} B(D)\big)$ contains a ball of radius ε_k.

Theorem 2.5. Let $k\geq 2$ be a natural number, $R>1$. Then there exists a subset $Q_k\subset\mathbb{R}^3$ with the property that any closed ball of radius R contains at least k elements of Q_k, and for any 2-colouring of Q_k there exists a ball B of radius R such that $B\cap Q_k$ is monochromatic.

Proof. Let $\underline{D}_k$ and P_k be the same as above, and set

$$Q_k = P_k\cup\Big(\mathbb{R}^3\setminus\bigcup_{D\in\underline{D}_k} B(D)\Big).$$

If B is any closed ball of radius R such that $B\notin\{B(D):D\in\underline{D}_k\}$, then, by Lemma 2.3, B contains at least one and hence infinitely many points of $\mathbb{R}^3\setminus\bigcup_{D\in\underline{D}_k} B(D)\subset Q_k$. If $B=B(D)$ for some $D\in\underline{D}_k$, then $|B\cap Q_k| = |B\cap P_k| = |D\cap P_k|=k$.

Let f be any coloring of Q_k by 2 colours. This induces a 2-colouring

of P_k; so, according to the definition of $\underline{D}_k$ (part (ii) of Theorem 2.1), there exists a $D \in \underline{D}_k$ such that $B(D) \cap Q_k = D \cap P_k$ is monochromatic. □

A point set $S \subseteq \mathbb{R}^n$ is called ε-<u>discrete</u> if $d(x,y) \geq \varepsilon$ for any two distinct elements $x, y \in S$.

<u>Theorem 2.5'</u>. For any natural number k, there exist $\varepsilon_k'' > 0$ and an ε_k''-discrete subset $Q_k' \subseteq \mathbb{R}^3$ with the property that any open unit ball contains at least k elements of Q_k', and for any 2-colouring of Q_k' one can find a closed unit ball B such that $B \cap Q_k'$ is monochromatic.

<u>Proof</u>. Let $\underline{D}_k$ and P_k satisfy the slightly stronger properties stated in Remark 2.4, and let S_k be an ε_k'-discrete subset of $\mathbb{R}^3 \setminus \bigcup_{D \in \underline{D}_k} B(D) = T_k$ such that every ball $B' \subseteq T$ whose radius is at least ε_k contains at least k elements of S_k. Set $Q_k' = P_k \cup S_k$. Then any open ball of radius R covers at least k elements of Q_k'; and for any 2-colouring of Q_k' there exist a $D \in \underline{D}_k$ such that all elements of $B(D) \cap Q_k' = D \cap P_k$ have the same colour. Changing the scale so that R becomes the unit distance, we obtain the result. □

<u>Theorem 2.5''</u>. For any natural numbers k and $d \geq 3$, there exist an $\varepsilon_{k,d} > 0$ and an $\varepsilon_{k,d}$-discrete subset $Q_{k,d} \subseteq \mathbb{R}^d$ with the property that any open unit ball contains at least k elements of $Q_{k,d}$, and for any 2-colouring of $Q_{k,d}$ one can find a closed unit ball B such that $B \cap Q_{k,d}$ is monochromatic.

<u>Proof</u>. We will prove by induction on d the following little stronger statement. There exist a $Q_{k,d} \subseteq \mathbb{R}^d$ with the required properties and a <u>finite</u> system $\underline{B}_{k,d}$ of closed unit balls such that for any 2-colouring of $Q_{k,d}$ one can find a $B \in \underline{B}_{k,d}$ for which all elements of $B \cap Q_{k,d}$ have the same colour.

For $d=3$, this follows from the proof of Theorem 2.5'.

Assume now that $Q_{k,d}$ and $\underline{B}_{k,d}$ have already been defined for some $d\geq 3$. Given any $B\in\underline{B}_{k,d}$, let B' denote a unit ball in $\mathbb{R}^{d+1}$ whose intersection with the hyperplane $\mathbb{R}^d$ is B. Let $\varepsilon>0$, and let $S_{k,d+1}$ denote a maximal ε-discrete subset of $\mathbb{R}^{d+1}\setminus\bigcup_{B\in\underline{B}_{k,d}} B'$. It is easy to see that, if ε is sufficiently small, then

$$Q_{k,d+1} = Q_{k,d}\cup S_{k,d+1}\ ,$$

$$\underline{B}_{k,d+1} = \{\ B' : B\in\underline{B}_{k,d}\ \}$$

satisfy all the conditions. □

Replacing each element $q\in Q_{k,d}$ by a unit ball centered at q, we obtain a k-fold covering of $\mathbb{R}^d$, which meets the requirements of Theorem 1. In this sense, Theorems 1 and 2.5" are 'dual' statements.

3. Proof of Theorem 2

Let $\underline{B}=\{B_i : i\in I\}$ be a k-fold covering of $\mathbb{R}^d$ with unit balls such that no point of $\mathbb{R}^d$ is contained in more than t members of $\underline{B}$. Assume without loss of generality that the balls are in general position. For any B_i, let bd B_i denote the surface of B_i. Let C_j $(j\in J)$ be the connected components of $\mathbb{R}^d \setminus \bigcup_{i\in I} \text{bd } B_i$, and define a hypergraph $H(\underline{B})=H$ in the following way. Set

$$V(H) = \{B_i : i\in I\},$$

$$E(H) = \{E_j : j\in J\} \quad \text{where} \quad E_j=\{B_i : C_j\subset B_i\in\underline{B}\}.$$

($V(H)$ and $E(H)$ denote the vertex set and the edge set of H, as usual.) The fact that $\underline{B}$ is a k-fold covering implies that

$$|E_j| \geq k \qquad \text{for every } j\in J. \tag{9}$$

Fix now a $j_o\in J$, and let E_j be any edge of H such that $E_j\cap E_{j_o}\neq\emptyset$. Then all elements (balls) belonging to E_j are contained in a ball B of radius 4 (around any point of C_{j_o}). Since no point of B is covered by more than t members of $\underline{B}$, we obtain

$$|\bigcup(E_j : E_j\cap E_{j_o}\neq\emptyset)| \leq \frac{t \text{ Vol } B}{\text{Vol } B_j} = t4^d.$$

On the other hand, it is easy to see that $t4^d$ balls cut the space into at most $(t4^d)^d$ different pieces, hence

$$|\{E_j : E_j\cap E_{j_o}\neq\emptyset\}| \leq t^d 4^{d^2} \quad \text{for any } j_o\in J. \tag{10}$$

The following result is an easy consequence of the Lovász Local Lemma (cf. [1], [6]).

Theorem 3.1. ([2], [3]) Let H be a hypergraph whose every edge has at least k elements. If every edge of H meets at most 2^{k-3} other edges, then there

exists a 2-colouring of $V(H)$ such that no edge is monochromatic. □

By (9) and (10), we can apply this result to our hypergraph H, provided that $t \leq 2^{(k/d)-2d-1}$. Thus, we obtain a colouring $f : V(H) \to \{R,G\}$ with 2 colours (say, red and green) such that every C_j is covered by both red and green balls. Consequently, $\underline{B} = \underline{B}_R \cup \underline{B}_G$,

$$\underline{B}_R = \{B_i \in \underline{B} : f(B_i)=R\},$$

$$\underline{B}_G = \{B_i \in \underline{B} : f(B_i)=G\}$$

is a decomposition of $\underline{B}$ into two coverings. □

Remark 3.2. The same proof shows that Theorem 2 remains valid (apart from the value of the constant) for every k-fold covering $\underline{B}=\{B_i : i \in I\}$ with balls satisfying $\frac{\inf_{i\in I} r(B_i)}{\sup_{i\in I} r(B_i)} > \varepsilon$ for some $\varepsilon>0$.

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