THE MAXIMUM NUMBER OF EMPTY CONGRUENT TRIANGLES DETERMINED BY A POINT SET

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Abstract

Let S be a set of n points in the plane and consider a family of (nondegenerate) pairwise congruent triangles whose vertices belong to S. While the number of such triangles can grow superlinearly in n — as it happens in lattice sections of the integer grid — it has been conjectured by Brass that the number of pairwise congruent *empty* triangles is only at most linear. We disprove this conjecture by constructing point sets with $\Omega(n \log n)$ empty congruent triangles.

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1 INTRODUCTION

Let S be a set of n points in the plane and P be another (smaller) set of points, called *pattern*. Establishing tight estimates on the maximum number of times a given pattern P can occur in S (under congruence, similarity, etc.) is a classical topic in discrete and combinatorial geometry, which was started by the following question of Erdős [5] (see also [2]): "At most how many times can the unit distance occur among a set of n points?"

Let $u(n)$ denote this maximum. In the same paper, Erdős proved that $u(n) = O(n^{3/2})$. This bound was later improved to $u(n) = O(n^{4/3})$ by Spencer, Szemerédi, and Trotter [9]. Erdős also showed that in a $\sqrt{n} \times \sqrt{n}$ section of the integer grid the same distance can occur $\Omega(n^{1+c/\log\log n})$ times, where $c > 0$ is an absolute constant. Therefore, we have $u(n) = \Omega(n^{1+c/\log\log n})$ [7]. The same bound holds for the triangular lattice. Both the upper and the lower bounds on the number of equal distances carry over (asymptotically) as upper and lower bounds on the maximum number

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of pairwise congruent triangles: each pair of points at distance r can be a side of length r in at most four congruent copies of a triangle. On the other hand, by the rotational symmetry of the triangular lattice, all point pairs that determine a given distance r can be extended on both sides to an equilateral triangle. As long as r is much smaller than the diameter of the n-element section S of the lattice, most of these pairs can be extended to two equilateral triangles within S. The same phenomenon occurs for triangles similar to any fixed triangle spanned by three points of the triangular lattice.

If we assume that the elements of an n-element point set S are in *convex position*, then the number of times that the same (unit) distance can occur among them is conjectured to be $O(n)$. Füredi established an upper bound of $O(n \log n)$ [6] (see also [3]), while in the best known construction, due to Edelsbrunner and Hajnal [4], the number of unit distance pairs in S is $2n-7$. For triangles, the problem has been essentially solved in this case: Pach and Pinchasi [8] proved that n points in convex position can span at most $4n$ congruent copies of a given triangle.

An interesting variant of the triangle problem is to consider *empty* triangles, that is, pairwise congruent triangles spanned by S such that none of them contains any element of S in its interior. This problem, posed by Brass, has some algorithmic motivation in connection with "window matching" [1]. Brass conjectured [2] that the maximum number of pairwise congruent empty triangles spanned by n points in the plane is $O(n)$. Here we disprove Brass' conjecture.

THEOREM 1 *For all n, there exist n-element point sets in the plane that span $\Omega(n \log n)$ pairwise congruent empty triangles.*

Note that it is still possible that for any *fixed* triangle T, the maximum number of empty congruent copies of T spanned by an n-element point set in the plane is only linear in n. This is the case when T is an *obtuse* or a *right-angled* triangle.

THEOREM 2 *For any obtuse or right-angled triangle T, there is a constant c_T such that the number of empty congruent copies of T spanned by an n-element point set in the plane is at most $c_T \cdot n$.*

2 PROOF OF THEOREM 1

The idea is simple: first construct a set of points S_0 with many (i.e., $\Omega(n \log n)$ pairwise congruent triples of collinear points — which can be viewed as degenerate empty congruent triangles. Then very slightly perturb this construction to obtain a set of points S so that these degenerate triangles become non-degenerate empty congruent triangles. The details are as follows.

Let $n = 3^k$. Consider k unit vectors $b_1, \dots, b_k$, and for $1 \le i \le k$, let β_i be the counterclockwise angle from the x-axis to b_i. We choose each β_i randomly — independently and uniformly from the interval $(0, \pi/2)$. Let $\lambda \in (0, 1)$ be fixed and let $a_i = \lambda b_i$.

Consider now all 3^k possible sums of these $2k$ vectors, a_i and b_i, $1 \le i \le k$, with coefficients 0 or 1, satisfying the condition that for each i, at least one of a_i or b_i has coefficient 0. and let S_0 be the set of their endpoints. Clearly, each triple of the form $(v,\, v + a_i,\, v + b_i)$ — where v is a subset sum that does not contain a_i or b_i — consists of collinear points. For such a triple, denote by $s_i(v)$ the segment whose endpoints are v and $v + b_i$, and by $p_i(v)$, $q_i(v)$, and $r_i(v)$ the points v, $v + a_i$, $v + b_i$ respectively. We say that the above triple is of type i, $i = 1, \ldots, k$. Obviously, for each i there are exactly 3^{k-1} triples of type i, therefore we have a total of

$$k3^{k-1} = \frac{n \log n}{3 \log 3} = \Omega(n \log n)$$

triples of collinear points. In fact, all these triples form degenerate congruent triangles in S_0. Denote by E the set of segments corresponding to these triples.

LEMMA 1 *There exist angles $\beta_1, \ldots, \beta_k$, such that*

(i) *S_0 consists of n distinct points;*

and

(ii) *if $v, u, v + b_i \in S_0$ are collinear (in this order), then $u = v + a_i$.*

Note that there may exist other triples of collinear points in S_0 (such as b_1, $a_1 + a_2$, b_2, for $\lambda = 1/2$). However, Lemma 1 does not apply to them.

Assume for a moment that the lemma holds. Let ϵ be the minimum distance between points $p \in S_0 \setminus \{p_i(v), q_i(v), r_i(v)\}$ and segments $s_i(v) \in E$ over all pairs v, i. By Lemma 1, $\epsilon > 0$. Now slightly modify the construction in the following way: instead of choosing a_i to be collinear with b_i, we slightly rotate λb_i counterclockwise from b_i through an angle of δ around their common origin. This modification is carried out at the same time for all vectors a_i, $i = 1, \ldots, k$ that appear in the construction. By continuity, there exists $\delta = \delta(\epsilon)$, so that each of the congruent degenerate triangles in the construction remains empty throughout this small perturbation.

It remains to prove Lemma 1. Write $[k] = \{1, \ldots, k\}$.

PROPOSITION 1 *Let $i \in [k]$ and $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ be fixed and nonzero. Then the probability that there exists $\mu \neq 0$ such that*

$$\sum_{j \in [k] \setminus \{i\}} \lambda_j b_j + \mu b_i = 0$$

is zero. In particular, the probability that $\sum_{i=1}^{k} \lambda_i b_i = 0$ is zero.

Proof. The probability that

$$\mu b_i = - \sum_{j \in [k] \setminus \{i\}} \lambda_j b_j,$$

for some μ, i.e., the vector b_i is parallel to $- \sum_{j \in [k] \setminus \{i\}} \lambda_j b_j$, is zero. $\square$

We can now prove (i): Assume that two given vector combinations yield the same point, that is

$$\sum_{i\in I_{11}} a_i + \sum_{i\in I_{12}} b_i = \sum_{i\in I_{21}} a_i + \sum_{i\in I_{22}} b_i, \tag{1}$$

where $I_{11} \cap I_{12} = \emptyset$, $I_{21} \cap I_{22} = \emptyset$. Write $B_{ij} = \sum_{k\in I_{ij}} b_k,\ \ i,j = 1,2$. Then (1) can be rewritten as

$$\lambda(B_{11} - B_{21}) + (B_{12} - B_{22}) = 0. \tag{2}$$

The above vector equation has some nonzero coefficient $\in \{\pm 1, \pm(1-\lambda)\}$ unless $I_{11} = I_{21}$ and $I_{12} = I_{22}$, that is, the two vector combinations are the same. Therefore, by Proposition 1, since there are only a finite number, $O(n^2)$, of pairs of vector combinations, if $\beta_1, \dots, \beta_k$ are chosen randomly, with high probability no two vector combinations yield the same point.

We continue with (ii): by a similar argument, there are only a finite number, $O(n^2 \log n)$, of triples of points of the form v_1, v_2, $v_3 = v_1 + b_i$, so it suffices to show that if $\beta_1, \dots, \beta_k$ are chosen randomly, with high probability a given triplet of points, other than those that exist by construction, consists of non-collinear points.

Assume therefore that the vector combinations v_1, v_2, $v_3 = v_1 + b_i$ are collinear in this order. Thus, for some $\mu \in (0,1)$, we have

$$v_2 = v_1 + \mu b_i. \tag{3}$$

Let

$$v_1 = \sum_{i\in I_{11}} a_i + \sum_{i\in I_{12}} b_i,\ \ v_2 = \sum_{i\in I_{21}} a_i + \sum_{i\in I_{22}} b_i,$$

where $I_{11} \cap I_{12} = \emptyset$, $I_{21} \cap I_{22} = \emptyset$, and $i \notin (I_{11} \cup I_{12})$. Write $B_{ij} = \sum_{k\in I_{ij}} b_k,\ \ i,j = 1,2$. Then (3) can be rewritten as

$$\lambda B_{21} + B_{22} = \lambda B_{11} + B_{12} + \mu b_i. \tag{4}$$

If $i \notin (I_{21} \cup I_{22})$, the coefficient of b_i in the resulting equation is $\mu \neq 0$, therefore by Proposition 1, for a random choice of angles, equation (4) holds with probability zero. If $i \in I_{22}$, the coefficient of b_i in the resulting equation is $(1-\mu) \neq 0$, and the same argument applies.

If $i \in I_{21}$, the coefficient of b_i in the resulting equation is $\lambda - \mu$. If $\lambda \neq \mu$, the coefficient of b_i is again nonzero, and the same argument applies. In the remaining case, $\lambda = \mu$, we either have (a) $I_{21} = I_{11} \cup \{i\}$ and $I_{22} = I_{12}$, which means $v_2 = v_1 + a_i$, that is, v_1, v_2, v_3 forms a triple of points collinear by construction, or (b) after reducing the terms $\lambda b_i = \mu b_i$, the resulting equation corresponds to two different vector combinations giving the same point, which holds with probability zero by Proposition 1. This concludes the proof of Lemma 1 and hence the proof of Theorem 1.

Remark. The ratio of two sides of our triangle is λ, so it can be chosen arbitrarily in $(0,1)$, and with a slight modification of our argument we can show that two sides can be chosen to be equal, that is, Theorem 1 holds with *isosceles* triangles. It would be interesting to know whether Theorem 1 or Theorem 2 (or none of them) holds with *equilateral* triangles.

3 PROOF OF THEOREM 2

Let T be an obtuse or right-angled triangle with vertices a, b, c, and angles α, β, γ respectively, with $\alpha \leq \beta \leq \gamma$. Let S be a set of n points, and consider all triangles determined by S congruent to T. Denote their number by m. Clearly, we can choose $m' = \lceil \frac{m\alpha}{4\pi} \rceil$ of these triangles congruent to T such that (i) all of them have the same orientation, and (ii) their corresponding sides determine an angle less than α. Let M' be the set of these triangles. Assume without loss of generality that the triangles in M' are clockwise oriented, that is, their vertices corresponding to a, b, and c follow each other in this clockwise order.

Define a directed graph on the points of S. We have an edge from x to y if and only if there is a triangle in M' such that x corresponds to a and y corresponds to c. This graph has one edge for each triangle in M'. Therefore, Theorem 2 is a direct consequence of the following:

PROPOSITION 2 *The out-degree of any vertex is at most one.*

Proof. Suppose that x has out-degree at least two. Let T_1 and T_2 be the corresponding triangles with vertices x, b_1, c_1 and x, b_2, c_2, see Fig. 1. By the choice of M', the angle $\widehat{c_1 x c_2}$ is less than α.

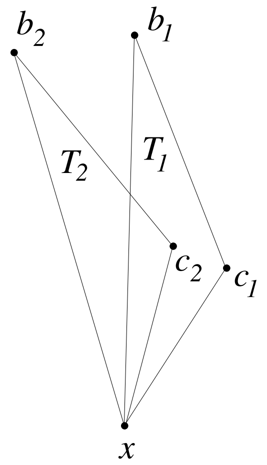

Figure 1: $c_2 \in T_1$.

Assume without loss of generality that $\widehat{c_1 x c_2}$ is oriented counterclockwise. Using that $\gamma \geq \pi/2$, we obtain that $c_2 \in T_1$, a contradiction. □

The argument shows that Theorem 2 holds with $c_T = 4\pi/\alpha$.

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