Unavoidable configurations in complete topological graphs

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Abstract

A *topological graph* is a graph drawn in the plane so that its vertices are represented by points, and its edges are represented by Jordan curves connecting the corresponding points, with the property that any two curves have at most one point in common. We define two canonical classes of topological complete graphs, and prove that every topological complete graph with n vertices has a canonical subgraph of size at least $c\log^{1/8} n$, which belongs to one of these classes. We also show that every complete topological graph with n vertices has a non-crossing subgraph isomorphic to any fixed tree with at most $c\log^{1/6} n$ vertices.

1 Introduction, results

A *topological graph* G is a graph drawn in the plane by Jordan curves, any two of which have at most one point in common. That is, it is defined as a pair $\{V(G), E(G)\}$, where $V(G)$ is a set of points in the plane and $E(G)$ is a set of simple continuous arcs connecting them so that they satisfy the following conditions:

1. no arc passes through any other element of $V(G)$ different from its endpoints;

2. any two arcs have at most one point in common, which is either a common endpoint or a proper crossing.

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$V(G)$ and $E(G)$ are the *vertex-set* and *edge set* of G, respectively. We say that H is a *(topological) subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Two topological graphs, G and H, are called *weakly isomorphic* if there is an incidence preserving one-to-one correspondence between $\{V(G), E(G)\}$ and $\{V(H), E(H)\}$ such that two edges of G intersect if and only if the corresponding edges of H do (see [C81]). If all edges of a topological graph are straight-line segments, then it is called a *geometric graph*. A geometric graph, whose vertices are in convex position, is called *convex*. Obviously, any two complete convex geometric graphs with m vertices are weakly isomorphic to each other, and to the convex geometric graph C_m, whose edge set consists of all sides and chords of a regular m-gon (see Fig. 1.).

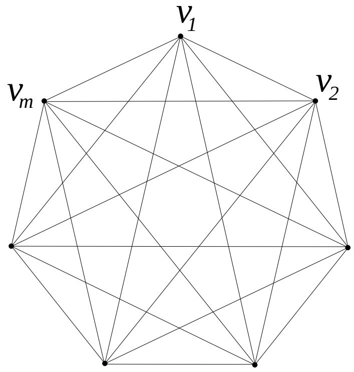

Fig. 1: *The convex geometric graph* C_m.

The fairly extensive literature on topological graphs focuses on very few special questions, and there is no standard terminology. For topological graphs, Erdős and Guy [EG73] (see also [AR88]) use the term "good drawings", while Gronau, Harborth, Mengersen, and Thürmann [GH90], [HM74], [HM90], [HT94] simply call them "drawings". For a complete topological graph, Ringel [R64] and Mengersen [M78] use the term "immersion". The most popular problems in this field are Turán's Brick Factory Problem [T77] (Zarankiewicz's Conjecture [G69] and other problems about *crossing numbers*, i.e., about the *minimum* number of crossings in certain drawings of a graph [PT98]) and Conway's Thrackle Conjecture [W71], [LPS97], [CN00] (and other problems about the *maximum* number of crossings in certain drawings of a graph [HM92]).

The systematic study of geometric graphs was initiated by Erdős, Avital–Hanani [AH66], Kupitz [K79], and Perles. (See [P99] and [PA95], Chapter 14, for the most recent surveys on the subject.) It is not hard to see that every *complete geometric graph* K_n of n vertices has a non-crossing subgraph isomorphic to any triangulation of a cycle of length n (cf. [GMPP91]). Consequently, K_n has a non-crossing subtree isomorphic to any fixed tree of n vertices. In particular, K_n has a non-crossing path of n vertices and a non-crossing matching of size $\lfloor n/2 \rfloor$. On the other hand, it is known that K_n has at least constant times $\sqrt{n}$ pairwise crossing edges [AEG94]. Our aim is to establish analogous results for topological graphs.

Theorem 1. *Every topological complete graph of n vertices has a non-crossing subgraph isomorphic to any fixed tree T with at most $c\log^{1/6} n$ vertices. In particular, it contains a non-crossing path with at least $c\log^{1/6} n$ vertices.*

According to a wellknown theorem of Erdős and Szekeres [ES35],[ES60], any set of n points in general position in the plane contains a subset with at least $c\log n$ elements which form the vertex set of a convex polygon. (Throughout this note, the letter c appearing in different assertions denote unrelated positive constants. The best known bound in the last statement is due Tóth and Valtr [TV98].) The Erdős-Szekeres Theorem can be reformulated, as follows.

Erdős-Szekeres Theorem. *Every complete geometric graph with n vertices has a complete geometric subgraph, weakly isomorphic to a convex complete graph C_m with $m \geq c\log n$ vertices.*

The situation is more complicated for *topological* graphs. In their study of topological complete graphs with m vertices and with the *maximum* possible number, $\binom{m}{4}$, of edge crossings, Harborth and Mengersen [HM92] found a drawing which contains no subgraph weakly isomorphic to C_5. We call this drawing, depicted in Figure 2, *twisted*, and denote it by T_m.

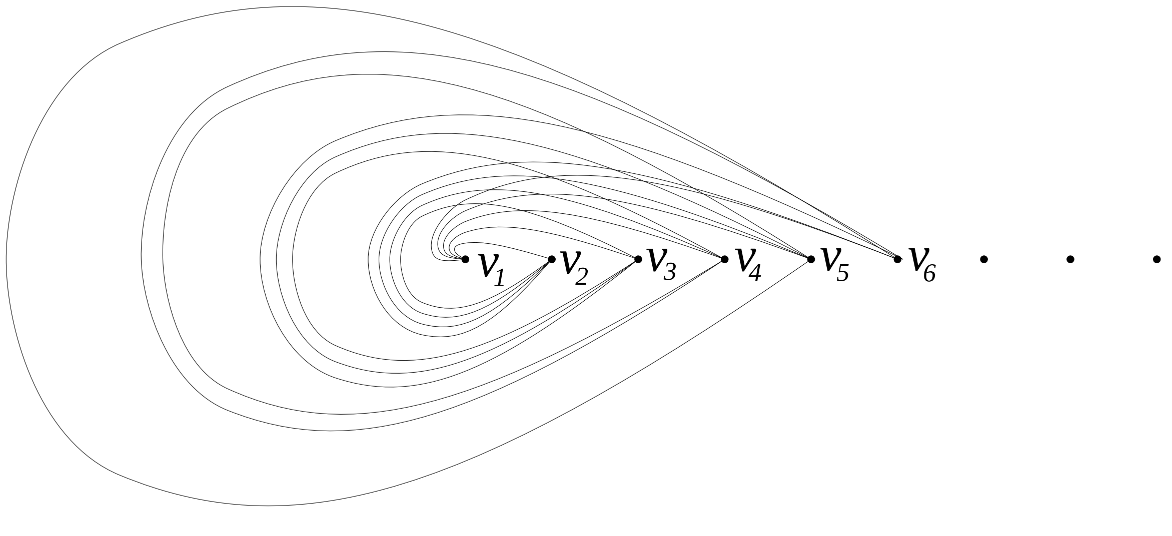

Fig. 2: *The twisted drawing T_m.*

We show that one cannot avoid *both* C_m and T_m in a sufficiently large complete topological graph.

Theorem 2. *Every complete topological graph with n vertices has a complete topological subgraph with $m \geq c\log^{1/8} n$ vertices, which is weakly isomorphic either to a convex complete graph C_m or* *to a twisted complete graph T_m.*

The arguments in the next section show that Theorem 2 easily implies a somewhat weaker version of Theorem 1, where $\log^{1/6} n$ is replaced by $\log^{1/8} n$.

2 Proofs

Before we turn to the proofs, we rephrase the definitions of convex and twisted complete topological graphs.

Definition 2.1. Let K_m be a complete topological graph on m vertices. If there is an enumeration of the vertices, $\{u_1, u_2, \ldots, u_m\}$, such that

(i) two edges, $u_i u_j$ $(i < j)$ and $u_k u_l$ $(k < l)$, cross each other if and only if $i < k < j < l$ or $k < i < l < j$, then K_m is called *convex*;

(ii) two edges, $u_i u_j$ $(i < j)$ and $u_k u_l$ $(k < l)$, cross each other if and only if $i < k < l < j$ or $k < i < j < l$, then K_m is called *twisted.*

Let K be a fixed complete topological graph with $n + 1$ vertices. The edges of K divide the plane into several cells, precisely one of which is unbounded. Without loss of generality, we can assume that there is a vertex $v_0 \in V(K)$ on the boundary of the unbounded cell. Otherwise, we can apply a stereographic projection to transform K into a drawing on a sphere, and then, by another projection, we can turn it into a topological graph weakly isomorphic to K, which satisfies the required property.

Consider all edges emanating from v_0, and denote their other endpoints by $v_1, v_2, \ldots, v_n$, in clockwise order. Color the triples $v_i v_j v_k$, $1 \leq i < j < k \leq n$ with eight different colors, according to the following rules. Each color is represented by a zero-one sequence abc of length 3. For any $i < j < k$,

1. set $a = 0$ if the edges $v_i v_j, v_0 v_k \in E(K)$ do not cross, and let $a = 1$ otherwise;

2. set $b = 0$ if the edges $v_i v_k, v_0 v_j \in E(K)$ do not cross, and let $b = 1$ otherwise;

3. set $c = 0$ if the edges $v_j v_k, v_0 v_i \in E(K)$ do not cross, and let $c = 1$ otherwise.

It is easy to see that the complete topological subgraph of K induced by the vertices v_0, v_i, v_j, v_k (as any other complete topological graph with 4 vertices) has at most one pair of crossing edges. Therefore, we have

Claim 2.2. None of the colors 011, 101, 110, or 111 can occur.

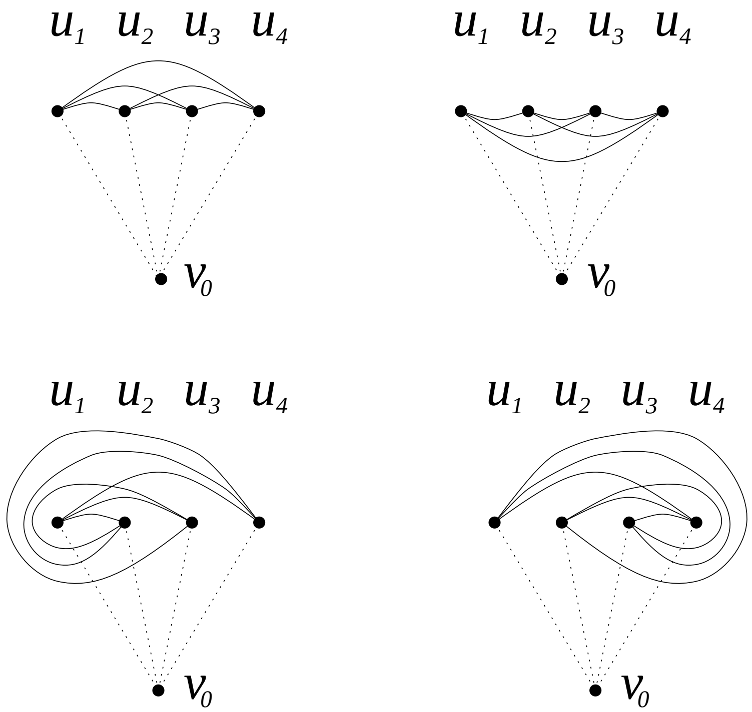

Fig. 3: *All triples are of type* 000, 010, 001, *and* 100, *respectively.*

Proof of Theorem 1: Let G be a topological complete graph with an $(n+1)$-element vertex set V. Use the same numbering, $v_0, v_1, \ldots, v_n$, of the vertices as in the previous section. For any $0 < i < j$, we say that v_i *precedes* v_j (in notation, $v_i \prec v_j$). As before, color the triples $v_i v_j v_k$ $(1 \le i < j < k \le n)$ with *four* colors, 000, 100, 010, and 001.

Claim 2.3. There exists an r-element subset $U := \{u_1, u_2, \ldots, u_r\} \subset \{v_1, v_2, \ldots, v_n\}$, $r \ge \sqrt{\log_4(n+1)}$ such that the triples $u_i u_j u_k$ and $u_i u_j u_l$ have the same color for any $i < j < k < l$.

Proof: The construction is recursive. Let $U_2 := \{v_1, v_2\}$ and $V_2 := V \setminus \{v_1, v_2\}$. Suppose that, for some $2 \le p < m$, we have already found two subsets $U_p = \{u_1, u_2, \ldots, u_p\}$ and $V_p \subset V$ with the properties

1. $u_1 \prec u_2 \prec \cdots \prec u_p$,
2. every element of U_p precedes all elements of V_p,
3. $|V_p| \ge \frac{|V_{p-1}|-1}{4^p}$.

Let u_{p+1} be the smallest element of V_p with respect to the ordering '$\prec$.' Since we used *four* colors for coloring the triples, there is a subset $W \subset V_p \setminus \{u_{p+1}\}$ with $|W| \ge (|V_p| - 1)/4^p$ such that, for each

$1 \le i \le p$, all triples $u_i u_{p+1} w$ ($w \in W$) have the same color. Let $U_{p+1} := U_p \cup \{u_{p+1}\}$ and $V_{p+1} := W$. An easy computation shows that this procedure can be repeated at least $\lceil \sqrt{\log_4 (n+1)} \rceil$ times. $\square$

Define the *type of an edge* $u_i u_j$ ($i < j < m$) as the color of a triple $u_i u_j u_k$ for any $k > j$. The type of $u_i u_r$ can be defined arbitrarily.

Let $G(100)$ and $G(001)$ denote the topological subgraphs of G consisting of all edges of type 100 and 001, resp., whose both endpoints belong to $U = \{u_1, u_2, \ldots, u_r\}$. The topological subgraph consisting of all other edges of G induced by U (of types 000 and 010) is denoted by G'.

Claim 2.4. Let $i < j < k < m$.

(i) If $u_i u_j$ and $u_j u_k$ belong to $G(100)$, then so does $u_i u_k$.

(ii) If $u_i u_j$ and $u_i u_k$ belong to $G(001)$, then so does $u_j u_k$.

Proof: If $u_i u_j$ is of type 100, it must cross both $v_0 u_k$ and $v_0 u_r$. If the type of $u_j u_k$ is also 100, it must cross $v_0 u_m$, too. Using the assumption that two edges that share an endpoint cannot have any other point in common, we obtain that $u_i u_k$ must cross $v_0 u_m$, which implies that its type is also 100 (see Fig. 3). This proves part (i). Part (ii) can be established similarly. $\square$

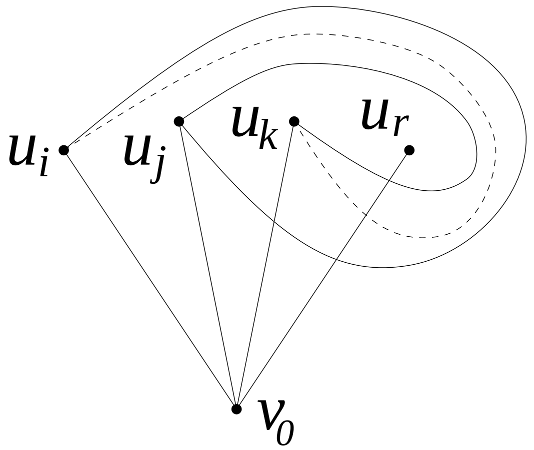

Fig. 4: *$u_i u_k$ must cross $v_0 u_r$.*

Claim 2.5. If $G(100)$, $G(001)$, or G' contains a complete subgraph of size $m := \lceil r^{1/3} \rceil$, then G has a non-crossing subgraph isomorphic to any tree of m vertices.

Proof: Suppose that $w_1 \prec w_2 \prec \ldots \prec w_m$ induce a complete (topological) subgraph in $G(100)$. It is easy to see that this subgraph is *twisted*, i.e., it is weakly isomorphic to T_m. Take an arbitrary tree T with m vertices. Starting at any vertex $z_1 \in V(T)$, explore all other vertices of T using *breadth-first search*. Let $z_1, z_2, \ldots, z_m$ be a numbering of the elements of $V(T)$, in the order in which they are encountered by the algorithm. Then the embedding $f(z_i) = w_i$ $(1 \le i \le r)$ maps T into a non-crossing copy of T in $G(100)$, and we are done.

The case when $G(001)$ contains a complete subgraph of size m can be treated similarly.

Assume now that G' has a complete subgraph with m vertices, $w_1 \prec w_2 \prec \ldots \prec w_r$. It is easy to see that if two edges, $w_i w_j$ $(i < j)$ and $w_k w_l$ $(k < l)$, cross each other, then we have $i < k < j < l$ or $k < i < l < j$. In other words, if two edges of this subgraph cross each other, the corresponding edges also cross in a drawing on the same vertex set, weakly isomorphic to the *convex* drawing C_m. Clearly, C_m contains a non-crossing copy of every tree with m vertices, so the same is true for G'. □

In view of the last claim, it remains to prove that at least one of $G(100)$, $G(001)$, and G' has a complete subgraph of size $m = \lceil r^{1/3} \rceil$. Suppose, in order to obtain a contradiction, that this is not the case.

If some element $u \in U = \{u_1, u_2, \ldots, u_r\}$ had at least $m-1$ larger neighbors in $G(001)$ with respect to the ordering $\prec$, then, by Claim 2.4 (ii), these neighbors together with u would induce a complete subgraph in $G(001)$, a contradiction.

Now we recursively construct a sequence $w_1 \prec w_2 \prec \ldots$ consisting of at least $r^{2/3}$ elements of U, which form an independent set in $G(001)$ (i.e., they induce a complete subgraph in $G(100) \cup G'$).

Let $W_0 := \emptyset$ and $U_0 := \{u_1, u_2, \ldots, u_r\}$. Suppose that, for some $p < m^{2/3}$, we have already found two subsets $W_p = \{w_1, w_2, \ldots, w_p\}$ and $U_p \subset \{u_1, u_2, \ldots, u_r\}$, such that

1. W_p is an independent set in $G(001)$,
2. every element of W_p precedes every element of U_p,
3. there is no edge between W_p and U_p,
4. $|U_p| \ge r - p(m-1)$.

If $U_p \neq \emptyset$, let w_{p+1} be the smallest element of U_p with respect to the ordering $\prec$, and set $W_{p+1} := W_p \cup \{w_{p+1}\}$. Let U_{p+1} denote the set obtained from U_p by the deletion of w_{p+1} and its larger neighbors. Clearly, we have $|U_{p+1}| \ge |U_p| - m + 1$, so that this procedure can be repeated at least $\lceil r^{2/3} \rceil$ times.

Define the *rank* of any element $w \in W := \{w_1, w_2, \ldots\}$, as the number of vertices of the longest monotone path (with respect to $\prec$) which ends at w in the subgraph of $G(100)$ induced by W. There is no element whose rank is at least $r^{1/3}$, otherwise, by Claim 2.4 (i), the vertices of the corresponding path would induce a complete subgraph of size at least m in $G(100)$, contradicting our assumptions.

Therefore, we can suppose that at least $r^{1/3}$ elements of W have the same rank. According to the definitions, these elements form an independent set in $G(100)$ as well as in $G(001)$. Thus, they induce a complete subgraph in G', again a contradiction. This proves Theorem 1.

Proof of Theorem 2: Let $m = \lceil r^{1/4} \rceil$ now. Just like in the proof of Theorem 1, one can show that either one of $G(100)$ and $G(001)$ has a complete subgraph of size m, or G' has a complete subgraph of size $s = \lceil r^{1/2} \rceil$. In cases when $G(100)$ or $G(001)$ has a complete subgraph of size m, we are done because that subgraphs are *twisted* (See Claim 2.2). Assume now that G' has a complete subgraph G'' of size s.

Define the *rank* of any vertex w of G'' as the length of the longest path $w_1 \prec w_2 \prec \ldots \prec w_k = w$ in G'' with all edges $w_i w_{i+1}$ of type 000.

Suppose first that all vertices have rank less than m. Since $m \geq \sqrt{s}$, there are m vertices of G'' of the same rank, so all edges among them are of type 010, therefore they induce a complete *convex* topological graph of m vertices.

So, we can assume that there is a vertex of rank m, that is, there are vertices of G'' $w_1 \prec w_2 \prec \ldots \prec w_m$ with all edges $w_i w_{i+1}$ of type 000.

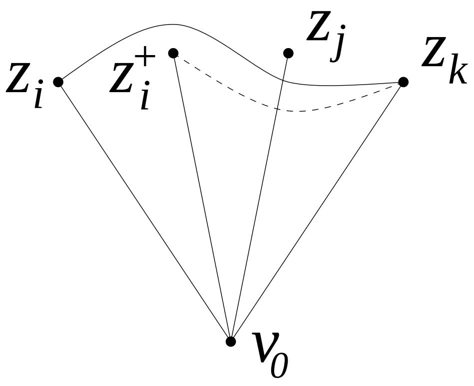

Fig. 5: $z_i^+ z_j$ *is also of type* 010.

Suppose that there is a triple $z_i \prec z_j \prec z_k \subset w_1 \prec w_2 \prec \ldots \prec w_m$ of color 010. Then $z_i z_j$ is of type 010. Choose a closest pair $z_i \prec z_j$ of type 010 on the path $w_1 \prec w_2 \prec \ldots \prec w_m$. The vertex next to z_i in the path is denoted by z_i^+. Clearly, $z_i^+ \neq z_j$ and $z_i^+ v_0$ has no crossing with the edge $z_i z_k$, because $z_i z_i^+$ has type 000. But then $z_i^+ \prec z_j \prec z_k$ should have color 010 too, so $z_i^+ z_j$ is also of type 010 contradicting the "minimality" of $z_i \prec z_j$. Consequently, all edges among $w_1 \prec w_2 \prec \ldots \prec w_m$ are of type 000, therefore they induce a complete *convex* topological graph of m vertices.

3 Concluding remarks

I. The following statement is a direct corollary of the first result in [PSS96].

Theorem 3.1. *Every complete topological graph of n vertices contains at least $c \log n / \log \log n$ pairwise crossing edges.*

II. Both C_m and T_m, the convex and the twisted topological graphs with m vertices, respectively, determine precisely $\binom{m}{4}$ edge crossings. Therefore, the following theorem of Harborth, Mengersen, and Schelp [HMS95] is an immediate consequence of Theorem 2.

Corollary 3.2 *For any positive integer m, there exists a smallest number $n(m)$ such that every complete topological graph with at least $n(m)$ vertices has a complete subgraph with m vertices and with $\binom{m}{4}$ crossings between its edges.*

In fact, for large values of m, Theorem 2 implies a better bound on the function $n(m)$ than the proof given in [HMS95].

III. Let F denote the graph obtained from a complete graph of 5 vertices by subdividing each of its edges with an extra vertex. Given a complete topological graph K_n of n vertices, define an abstract graph G. Let the vertex set of G consist of $\lfloor n/2 \rfloor$ edges of K_n, no two of which share an endpoint. Let two vertices, $e, e' \in E(K_n)$, be joined by an edge of G if and only if e and e' cross each other. It is easy to see that G does not contain F as an induced subgraph (see e.g. [EET76]).

It follows from a theorem of Erdős and Hajnal [EH89] that, if a graph with m vertices does not contain some fixed induced subgraph F, then it must have either an empty or a complete subgraph with at least $e^{c\sqrt{\log m}}$ vertices, where $c > 0$ is a constant depending on F. Putting these two facts together, we obtain

Corollary 3.3. *Any topological complete graph with n vertices has at least $e^{c\sqrt{\log n}}$ edges that are either pairwise disjoint or pairwise crossing.*

This suggests that the bounds in Theorems 1 and 3.2 are far from being optimal. We conjecture that both estimates can be replaced by n^{δ}, for some $\delta > 0$. As was pointed out in the Introduction, this holds for geometric graphs.

IV. In the case of *geometric* graphs, one can introduce several partial orderings on the set of edges (cf. [PT94], [PA95]). This allows us to apply Dilworth's Theorem in place of Ramsey's Theorem, to find much larger homogeneous substructures.

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