Uniformly distributed distances – a geometric application of Janson's inequality

János Pach[*]
City College, CUNY and Courant Institute, NYU

Joel Spencer
Courant Institute, NYU

Abstract

Let $d_1 \leq d_2 \leq \ldots \leq d_{\binom{n}{2}}$ denote the distances determined by n points in the plane. It is shown that $\min \sum_i (d_{i+1} - d_i)^2 = O(n^{-6/7})$, where the minimum is taken over all point sets with minimal distance $d_1 \geq 1$. This bound is asymptotically tight.

1 A problem of Erdős on distance gaps

Consider n points in the plane with minimum distance at least one. List the $m = \binom{n}{2}$ distances between them in increasing order: $d_1 \leq d_2 \leq \ldots \leq d_m$. The numbers $d_{i+1} - d_i$ will be called *distance gaps.* Erdős raised the following problem. Determine or estimate

$$f(n) = \min \sum_{1 \leq i < \binom{n}{2}} (d_{i+1} - d_i)^2, \tag{1}$$

where the minimum is taken over all n-element point sets with minimum distance one. In particular, he asked if this sum can be made arbitrarily small.

This choice of function to be minimized may at first appear to be capricious. Suppose, however, that we fix $d_1 = 1$ (which we may assume by a

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simple scaling argument) and the *diameter* $D = d_m$. Then if the numbers d_i could be chosen arbitrarily, $\sum_{1\le i<m}(d_{i+1}-d_i)^2$ would attain its minimum when the d_i are equally spaced. We have

$$\sum_{1\le i<m}(d_{i+1}-d_i)^2 \ge (m-1)^2\left(\frac{D-1}{m-1}\right)^2 = \Omega(D^2/n^2). \tag{2}$$

The geometric constraints make it impossible to achieve perfectly even spacing.

It was shown in [EMPS91] that the number of distances belonging to the interval $[D-1, D]$ is at most $O(D^{3/2})$. So even if these distances are evenly spaced, the gaps between them contribute at least $\Omega(D^{-3/2})$ to the sum $\sum(d_{i+1}-d_i)^2$. Combining this with (2), we obtain

$$f(n) \ge \min_D \left(\Omega(D^2/n^2) + \Omega(D^{-3/2})\right) = \Omega(n^{-6/7}). \tag{3}$$

Our main objective is to show that this bound is asymptotically tight.

Theorem 1. *There exists a set of n points in the plane with minimum distance at least one such that the distances d_i determined by them satisfy*

$$\sum_{1\le i<\binom{n}{2}}(d_{i+1}-d_i)^2 \le cn^{-6/7},$$

where c is an absolute constant.

It follows from (3) that the diameter D of such a point set must be approximately $n^{4/7}$ and that the interpoint distances must be fairly uniformly distributed in the interval $[1, D]$. Our construction described in Section 4 is based on a *Poisson process.* For the analysis, we use the "No Bonds Theorem" (Section 3), which is a continuous version of Janson's Inequality (Section 2). Sections 5–6 contain the details of the proof of Theorem 1.

2 Janson's inequality

Let X be a finite underlying set, and let P be a random subset of X, whose elements are chosen independently with probability $\Pr[x \in P] = p_x$. Let $\{S_i : i \in I\}$ be a system of subsets of X, and let A_i denote the event that $S_i \subseteq P$. If $S_i \cap S_j = \emptyset$ then A_i and A_j are independent. Let

$$\nu = \sum \Pr[A_i \wedge A_j],$$

where the sum is taken over all unordered pairs $i \neq j$ with $S_i \cap S_j \neq \emptyset$, and let

$$M = \prod_{i \in I} \Pr[\overline{A}_i] = \prod_{i \in I} (1 - \Pr[A_i]).$$

Janson's Inequality [J90]. *If* $\Pr[A_i] < \varepsilon$ *for every* $i \in I$*, then*

$$M \leq \Pr[\wedge_{i \in I} \overline{A}_i] \leq M e^{\nu/(2-2\varepsilon)}.$$

Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$. We apply Janson's Inequality in the specific case when $X = V(G)$ and $\{S_i : i \in I\} = E(G)$, i.e., $|S_i| = 2$ for every i. Then

$$M = \prod_{\{x,y\} \in E(G)} (1 - p_x p_y),$$

and ν is the expected number of "vees" (paths of length two) in $G|_P$, the subgraph of G induced by P.

Corollary. *Assume that for every edge* $\{x, y\}$ *of a graph* G*,* $p_x p_y \leq \varepsilon$*. Then,*

$$M \leq \Pr[G|_P \text{ is empty}] \leq M e^{\nu/(2-2\varepsilon)}.$$

Note that if $\max_{x \in V(G)} p_x$ is small, then M can be well approximated by

$$\prod_{\{x,y\} \in E(G)} e^{-p_x p_y} = e^{-\mu},$$

where

$$\mu = \sum_{\{x,y\} \in E(G)} p_x p_y$$

is the expected number of edges in $G|_P$.

3 Poisson processes

Now we extend the Corollary to the continuous case to be used in the sequel.

Let $X \subset \mathbb{R}^2$ be a bounded Jordan measurable set, and let $\sim$ be a symmetric binary relation (graph) on X such that $\{(x, y) : x \sim y\}$ is a Jordan measurable subset of $\mathbb{R}^2 \times \mathbb{R}^2$. If $x \sim y$ for some $x, y \in X$, then they are said to form a *bond*. Furthermore, let φ be a countably additive finite measure on the Borel subsets of X, defined by

$$\varphi(Y) = \int_Y \rho(x)dx,$$

where $\rho : X \to [0, +\infty)$, the *density function* of φ, is Riemann integrable.

Let $P \subset X$ be a random multiset given by a *Poisson process* associated with the measure φ. More precisely, P is a random variable whose values are almost surely unordered i-tuples of X (possibly with repetition) for some non-negative integer i such that

$$\Pr[|P| = i] = \frac{\varphi^i(X)}{i!} \cdot e^{-\varphi(X)} \quad (i = 0, 1, 2, \ldots),$$

and for a fixed i, P can be obtained by selecting i points from X independently with uniform distribution with respect to φ and disregarding their order. It is now easy to check that for any Borel set $Y \subseteq X$,

$$\Pr[|P \cap Y| = i] = \frac{\varphi^i(Y)}{i!} \cdot e^{-\varphi(Y)} \quad (i = 0, 1, 2, \ldots), \tag{4}$$

where $|P \cap Y|$ counts the number of points of P belonging to Y with multiplicities. In particular, the expected value of $|P \cap Y|$ is equal to $\varphi(Y)$. Moreover, if Y_1 and Y_2 are disjoint then $|P \cap Y_1|$ and $|P \cap Y_2|$ are independent random variables.

Let B and V denote the number of bonds and the number of "vees" formed by the elements of P, respectively, i.e.,

$$B = |\{\{x, y\} : x, y \in P, x \sim y\}|,$$
$$V = |\{(x, \{y, z\}) : x, y, z \in P, y \neq z, \text{ and } x \sim y, x \sim z\}|,$$

and set $\mu = E[B], \nu = E[V]$.

Theorem 2. (No Bonds Theorem) *Let $P \subset X$ be a random multiset obtained by a Poisson process associated with the measure φ, and let B denote the number of bonds between elements of P.*

Then, with the above notations and assumptions, we have

$$e^{-\mu} \leq \Pr[B = 0] \leq e^{-\mu+\nu}.$$

Proof: For a fixed n, place a mesh $X_n = \{(i/n, j/n) : i, j \in \mathbb{Z}\}$ of sidelength n in the plane, and let P_n be a random multisubset of X_n obtained by the Poisson process associated with the measure

$$\varphi_n(Y) = \sum_{\left(\frac{i}{n},\frac{j}{n}\right)\in Y} \varphi\left(\left[\frac{i}{n},\frac{i+1}{n}\right]\times\left[\frac{j}{n},\frac{j+1}{n}\right]\right)$$

for any finite subset $Y \subseteq X_n$. If Y consists of a single element $x \in X_n$ then, by (4),

$$\Pr[x \in P_n] = \sum_{i=1}^{\infty} \frac{\varphi_n^i(x)}{i!} \cdot e^{-\varphi_n(x)} = 1 - e^{-\varphi_n(x)}.$$

Moreover, the events $x \in P_n$ are independent for all $x \in X_n$.

Let G_n denote the graph on the vertex set X_n, whose two points $x, y \in X_n$ are joined by an edge if and only if $x \sim y$. Furthermore, let B_n denote the number of bonds formed by elements of P_n, where each bond is counted only *once*. It follows from the Corollary in the last section that

$$M_n \leq \Pr[B_n = 0] \leq M_n e^{\nu_n}, \tag{5}$$

where

$$M_n = \prod_{\{x,y\}\in E(G_n)} \left(1 - \Pr[x \in P_n]\Pr[y \in P_n]\right),$$

$$\nu_n = \sum_{\substack{(x,\{y,z\}):\\ xy,\, xz\, \in\, E(G_n),\, y\neq z}} \Pr[x \in P_n]\Pr[y \in P_n]\Pr[z \in P_n].$$

Notice that, as n tends to infinity,

$$\Pr[x \in P_n] = 1 - e^{-\varphi_n(x)} = 1 - \exp\left(-\int_{x+[0,1/n]^2} \rho(x)dx\right) \to 0$$

uniformly for all x. Since ρ is Riemann integrable,

$$\begin{aligned}\lim_{n\to\infty} M_n &= \lim_{n\to\infty} \exp\left(-(1+o(1))\cdot\sum_{\{x,y\}\in E(G_n)} \varphi_n(x)\varphi_n(y)\right)\\ &= \exp\left(-\frac{1}{2}\int_{x\in X}\int_{\substack{y\,\in\, X\\ y\,\sim\, x}} \rho(x)\rho(y)dydx\right) = e^{-\mu},\end{aligned}$$

and

$$\lim_{n\to\infty} \nu_n = \lim_{n\to\infty} (1+o(1)) \cdot \sum_{\substack{(x,\{y,z\}) : \\ xy,xz \in E(G_n), y \neq z}} \varphi_n(x)\varphi_n(y)\varphi_n(z)$$

$$= \frac{1}{2} \int_{x\in X} \int_{\substack{y \in X \\ y \sim x}} \int_{\substack{z \in X \\ z \sim x}} \rho(x)\rho(y)\rho(z) dz dy dx = \nu.$$

It remains to show that

$$\lim_{n\to\infty} \Pr[B_n = 0] = \lim_{n\to\infty} \Pr[B = 0],$$

and then the theorem follows from (5).

By definition, $\varphi_n(X_n) = \varphi(X) = \int_X \rho(x)dx$. Hence, in view of (4),

$$\Pr[B_n = 0] = \sum_{i=0}^{\infty} \Pr\left[B_n = 0 \mid |P_n| = i\right] \Pr\left[|P_n| = i\right]$$

$$= \sum_{i=0}^{\infty} \Pr\left[B_n = 0 \mid |P_n| = i\right] \cdot \frac{\varphi^i(X)}{i!} \cdot e^{-\varphi(X)},$$

$$\Pr[B = 0] = \sum_{i=0}^{\infty} \Pr\left[B = 0 \mid |P| = i\right] \cdot \frac{\varphi^i(X)}{i!} \cdot e^{-\varphi(X)}.$$

It suffices to verify that for any fixed i,

$$\lim_{n\to\infty} \Pr\left[B_n = 0 \mid |P_n| = i\right] = \Pr\left[B = 0 \mid |P| = i\right]. \tag{6}$$

Restricting P_n (and P) to the case $|P_n| = i$ (and $|P| = i$), their points are distributed on X_n (on X) independently from each other and uniformly with respect to φ_n (φ, respectively). Thus, $\Pr\left[B = 0 \mid |P| = i\right]$ can be expressed as $\int \rho(x_1)\dots\rho(x_i)dx_1\dots dx_i$ over a Jordan measurable subset of $X \times \dots \times X \subseteq \mathbb{R}^{2i}$, and $\Pr\left[B_n = 0 \mid |P_n| = i\right]$ will approximate it with arbitrary precision, as $n \to \infty$. This proves (6), and hence the theorem. □

4 Outline of the upper bound construction

The aim of this section is to sketch a probabilistic construction for the proof of Theorem 1. The details will be worked out in the next two sections.

As we have indicated in the last paragraph of Section 1, any $O(n)$-element point set that satisfies the inequality in Theorem 1 must have diameter $\Theta(n^{4/7})$. All of our points will be chosen from the closed disk of radius $n^{4/7}$ around the origin $(0,0) \in \mathbf{R}^2$.

The construction consists of three parts (see Figure 1).

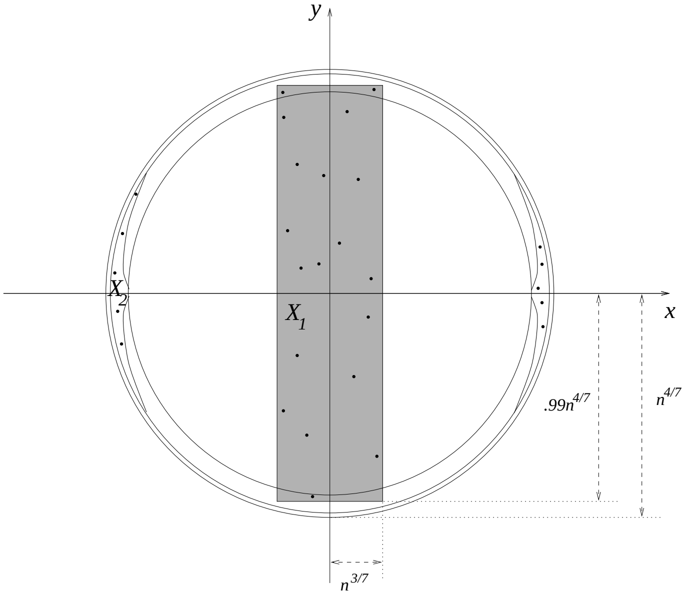

Figure 1

Step 1. Define a point set P_1 by a Poisson process on the rectangle

$$X_1 = \{(x,y) : |x| \leq n^{3/7}, |y| \leq 0.99n^{4/7}\}$$

with constant density function $\rho(x,y) \equiv \varepsilon$, where ε is a small positive number (Subsection 5.1). If two elements of P_1 are at distance less than one, delete both of them, giving a set P_1^* (Section 6). The smallness of ε, we can take $\varepsilon = 10^{-3}$ for definiteness, is needed only to assure that not too many vertices are deleted.

Step 2. Define a point set P_2 by a Poisson process with constant density ε on the region X_2 given by the polar coordinates

$$\{(r,\theta) : 0.9n^{4/7} < r < n^{4/7} - 1, \min(|\theta|, |\theta - \pi|) < 0.5(n^{4/7} - r)^{-1/4}\}.$$

As in the previous step, delete every pair of points whose distance is less than one, and denote the resulting set by P_2^* (Subsection 5.2 and Section 6).

Step 3. We give an explicit construction P_3 of points on the circle of radius $n^{4/7}$,

$$\{(x,y) : |x|^2 + |y|^2 = n^{8/7}\},$$

with minimum distance at least one. (Subsection 5.3.)

Note that the distance between any two points belonging to different P_i's ($i = 1,2,3$) is larger than one. We will show that the number of points in $P = P_1 \cup P_2 \cup P_3$ is almost surely $O(n)$ and, with probability at least $1/2$, the sum of the squares of the distance gaps determined by them satisfies $\sum(d_{i+1} - d_i)^2 = O(n^{-6/7})$. In Section 6, we handle the somewhat technical problem of showing that this bound also holds for the slightly smaller $P_1^* \cup P_2^* \cup P_3$, which is our final point set.

First we need some terminology. Let us classify all real numbers between 1 and $D = 2n^{4/7}$ (the diameter of our construction). A number t is said to be

$$\begin{array}{llrcl} \textit{moderate} & \text{if} & 1 \leq & t & \leq 1.96n^{4/7}, \\ \textit{large} & \text{if} & 1.96n^{4/7} < & t & \leq D - 3, \\ \textit{extra large} & \text{if} & D - 3 < & t & \leq D. \end{array}$$

For every j, $1 \leq j < D = 2n^{4/7}$ and for all $k = 0,1,2,\ldots$, divide the interval $[j, j+1)$ into 2^k equal subintervals of length 2^{-k}. That is, let

$$I_{jkl} = [j + (l-1)2^{-k}, j + l2^{-k}), \quad 1 \leq l \leq 2^k.$$

In the sequel, these intervals will be called *canonical.* Obviously, any interval $I \subset [1, D)$ with $|I| \leq 1$ has a canonical subinterval

$$I_{jkl} \subseteq I \quad \text{satisfying} \quad |I_{jkl}| \geq \frac{|I|}{4},$$

where $|I|$ denotes the *length* of I.

We say that an interval I is *empty* if our point set P has no pair of points whose distance belongs to I. Our goal is to show that with large probability the sum of the squares of the distance gaps determined by P is small. We will see that our construction has no distance gaps longer than one. Since for any two consecutive distances, d_i and d_{i+1}, (d_i, d_{i+1}) is an empty interval, we have

$$\sum_{1 \leq i < \binom{n}{2}} (d_{i+1} - d_i)^2 \leq 4^2 \cdot \sum_{\substack{I_{jkl} \text{ empty} \\ \text{canonical}}} |I_{jkl}|^2. \tag{7}$$

That is why in the rest of the paper we will concentrate on establishing upper bounds for the expected value

$$E\,[\sum_{\substack{I_{jkl}\text{ empty}\\ \text{canonical}}} |I_{jkl}|^2] = \sum_{I_{jkl}\text{ canonical}} |I_{jkl}|^2 \Pr[I_{jkl}\text{ is empty}]. \tag{8}$$

In fact, to estimate the terms $\Pr[I_{jkl}\text{ is empty}]$ from above, it will be sufficient to take into account the moderate distances generated by P_1^*, the large distances generated by P_2^*, and the extra large distances generated by P_3 (see Subsections 5.1, 5.2, and 5.3, repectively).

5 Bounding the distance gaps

5.1. Moderate distances. First we estimate the part of (8), where the sum is taken over all canonical intervals I_{jkl} for which j is *moderate*, and the point set is the set P_1 defined by a Poisson process in the rectangle X_1, as described in Step 1 above. The expected number of elements of P_1 is $\varepsilon\ \mathrm{Area}(X_1) = 3.96\varepsilon n = \Theta(n)$.

Fix a canonical interval $I = I_{jkl}$ with $1 \le j \le 1.96n^{4/7}$. We want to apply the No Bonds Theorem to the case when a pair of points $x, y \in P_1$ forms a bond if $|y - x| \in I$. Then the expected number of bonds and the expected number of "vees" satisfy

$$\mu = \frac{1}{2}\int_{x\in X_1}\int_{\substack{y\,\in\,X_1\\ |y-x|\,\in\,I}} \varepsilon^2 dydx = \Theta(\varepsilon^2 n \min(j, n^{3/7})2^{-k}),$$

$$\nu \le \int_{x\in X_1}\int_{\substack{y\,\in\,X_1\\ |y-x|\,\in\,I}}\int_{\substack{z\,\in\,X_1\\ |z-x|\,\in\,I}} \varepsilon^3 dzdydx = \Theta(\varepsilon^3 n \min(j^2, n^{6/7})2^{-2k}).$$

Thus, by the No Bonds Theorem,

$$\Pr[I\text{ is empty}] \le e^{-\mu/2} \le \exp\left(-\Omega(\varepsilon^2 n \min(j, n^{3/7})2^{-k})\right),$$

whenever $|I| = 2^{-k} \le 1/\min(j, n^{3/7})$. In particular,

$$\Pr[I\text{ is empty}] \le \exp\left(-\Omega(\varepsilon^2 n^{3/7})\right) \tag{9}$$

holds when $|I| = 2^{-k}$ is roughly equal to $n^{-4/7}$, and hence (9) is valid for $k \le (4/7)\log_2 n$.

Therefore, the total contribution to (8) of all canonical intervals I_{jkl} for which j is moderate,

$$\sum_{j \text{ moderate}} |I_{jkl}|^2 \Pr[I_{jkl} \text{ is empty}]$$

$$= \sum_{j=1}^{1.96n^{4/7}} \sum_{k=0}^{\frac{4}{7}\log n} \sum_{l=1}^{2^k} 2^{-2k} \Pr[I_{jkl} \text{ is empty}]$$

$$+ \sum_{j=1}^{1.96n^{4/7}} \sum_{k>\frac{4}{7}\log n} \sum_{l=1}^{2^k} 2^{-2k} \Pr[I_{jkl} \text{ is empty}]$$

$$\leq 1.96n^{4/7} \left(\frac{4}{7}\log n + 1\right) \exp\left(-\Omega(\varepsilon^2 n^{3/7})\right)$$

$$+ \sum_{j=1}^{1.96n^{4/7}} \sum_{k\geq\frac{4}{7}\log n} 2^{-k} \exp\left(-\Omega(\varepsilon^2 n \min(j, n^{3/7}) 2^{-k})\right)$$

$$\leq \exp(-\Omega(n^{3/7})) + \sum_{j=1}^{1.96n^{4/7}} O\left(\frac{1}{\varepsilon^2 n \min(j, n^{3/7})}\right) = O(n^{-6/7}).$$

Here we used the fact that the sum over k can be majorated by constant times its largest term.

5.2. Large distances. One can apply a similar argument to estimate the contribution of those canonical intervals $I = I_{jkl}$ to (8), for which j is *large*. Now we have to consider the Poisson process P_2 described in Step 2 (see Section 4). Clearly, the expected number of points of P_2 is equal to

$$\varepsilon \text{Area}(X_2) = 2\varepsilon \int_{0.9n^{4/7}}^{n^{4/7}-1} 0.5(n^{4/7} - r)^{-1/4} r dr = \Theta(\varepsilon n).$$

Fix a canonical interval $I = I_{jkl}$ with $1.96n^{4/7} < j \leq D - 3 = 2n^{4/7} - 3$, and say that a pair of points forms a *bond* if their distance belongs to I. Just like in the previous subsection, some routine calculation shows that the expected number of bonds is

$$\mu = \Omega(\varepsilon^2 n^{6/7} (D - j)^{5/4} 2^{-k}).$$

The No Bonds Theorem now implies that

$$\Pr[I \text{ is empty}] \leq e^{-\mu/2} \leq \exp\left(-\Omega(\varepsilon^2 n^{6/7} (D-j)^{5/4} 2^{-k})\right),$$

whenever $|I| = 2^{-k} \le n^{-4/7}$. This yields

$$\Pr[I \text{ is empty}] \le \exp\left(-\Omega(\varepsilon^2 n^{2/7})\right),$$

for every $k \le (4/7)\log_2 n$.

Therefore, the total contribution to (8) of all canonical intervals I_{jkl} for which j is large,

$$\sum_{j \text{ large}} |I_{jkl}|^2 \Pr[I_{jkl} \text{ is empty}]$$

$$\le \sum_{j=1.96n^{4/7}}^{2n^{4/7}-3} \sum_{k=0}^{\frac{4}{7}\log n} \sum_{l=1}^{2^k} 2^{-2k} \Pr[I_{jkl} \text{ is empty}]$$

$$+ \sum_{j=1.96n^{4/7}}^{2n^{4/7}-3} \sum_{k\ge\frac{4}{7}\log n} \sum_{l=1}^{2^k} 2^{-2k} \Pr[I_{jkl} \text{ is empty}]$$

$$\le 0.04n^{4/7}\left(\frac{4}{7}\log n + 1\right)\exp\left(-\Omega(\varepsilon^2 n^{2/7})\right)$$

$$+ \sum_{j=1.96n^{4/7}}^{2n^{4/7}-3} \sum_{k\ge\frac{4}{7}\log n} 2^{-k}\exp\left(-\Omega(\varepsilon^2 n^{6/7}(D-j)^{5/4}2^{-k})\right)$$

$$\le \exp(-\Omega(n^{2/7})) + \sum_{j=1.96n^{4/7}}^{D-3} O\left(\frac{1}{\varepsilon^2 n^{6/7}(D-j)^{5/4}}\right) = O(n^{-6/7}).$$

5.3. Extra large distances. Here we present an explicit (i.e., non-probabilistic) construction of a set P_3 of $O(n^{4/7})$ points on the circle of radius $n^{4/7}$ centered at the origin O. Let

$$P_3 = \{p_s : 0 \le s \le n^{4/7}/2\} \cup \{q_t : 0 \le t \le n^{4/7}/2\},$$

where, using polar coordinates (r, θ),

$$p_s = (n^{4/7}, 2sn^{-4/7}), \quad q_t = (n^{4/7}, \pi + 2t(n^{-4/7} + 4n^{-8/7})).$$

For any $t \ge s$, the clockwise angle $p_s O q_t$ is

$$\pi - 2(t-s)n^{-4/7} - 8tn^{-8/7}.$$

Thus, the angles p_sOq_t $(0 \le s \le t)$ are fairly densely distributed in the interval $(\pi - 1/2, \pi)$; every closed subinterval of length $8n^{-8/7}$ contains at least one of them. Consider the distances corresponding to these angles. It is easy to see that they divide $[2n^{4/7} - 3,\ 2n^{4/7}]$ into subintervals of length at most $15n^{-6/7}$. Consequently, the sum of the squares of the distance gaps in this interval is at most $45n^{-6/7}$, and

$$\sum_{\substack{I_{jkl} \text{ empty} \\ j \text{ extra large}}} |I_{jkl}|^2 = O(n^{-6/7}). \tag{10}$$

6 Excluding point pairs at distance less than one

So far we have focused on bounding (8), the expected value of the sum of the squares of the distance gaps determined by our random construction $P_1 \cup P_2 \cup P_3$. The deterministic part of the construction, P_3, satisfies the requirement that it has no two points at distance less than one, and it is also true that the distance between any two points belonging to different P_i's is at least one. However, some of the distances induced by P_1 and P_2 may be shorter than one.

In this section, we will deal with this problem. Let P_i^* denote the point set obtained from P_i by deleting every point $x \in P_i$ for which there is another point $y \in P_i$ with $|y - x| < 1$ $(i = 1, 2)$. In fact, instead of (8), we need an upper bound on

$$\sum_{I_{jkl} \text{ canonical}} |I_{jkl}|^2 \Pr{}^*[I_{jkl} \text{ is empty}],$$

where $\Pr^*$ denotes the probability that a given condition is satisfied for $P_1^* \cup P_2^* \cup P_3$, i.e., after the deletions have been carried out.

Let us concentrate on *moderate* distances, i.e., on the set P_1^*. Fix again a canonical interval $I = I_{jkl}$ with $1 \le j \le 1.96n^{4/7}$. A pair of points $\{x, y\} \subseteq P_1$ is said to form a bond if $|y - x| \in I$. A bond is called *bad* if there exists $z \in P_1$ whose distance from x or from y is less than one. In this case, $\{x, y\} \not\subseteq P_1^*$. Two bonds $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are *separated* if their distance is at least 2. Let μ and ν denote the same as in 5.1.

Lemma 6.1 *If $k \ge (4/7)\log_2 n$, then the probability that there are at least $\mu/10$ pairwise separated bonds is larger than $1 - e^{-\mu/8}$.*

Proof: We prove the stronger statement that the probability that there is a *maximal* family $\mathcal{F}$ of pairwise separated bonds with $|\mathcal{F}| < \mu/10$ is smaller than $e^{-\mu/8}$.

Indeed, this probability is at most

$$\sum_{i<\frac{\mu}{10}} \sum_{|\mathcal{F}|=i} \Pr[\mathcal{F} \text{ is maximal family of separated bonds}].$$

Since the expected number of i-tuples of bonds is $\mu^i/i!$, the above sum cannot exceed

$$\sum_{i<\frac{\mu}{10}} \frac{\mu^i}{i!} \cdot \max_{|\mathcal{F}|<\frac{\mu}{10}} \Pr[\mathcal{F} \text{ is maximal} \mid \mathcal{F} \text{ consists of bonds}]. \tag{11}$$

The conditional probabilities in the last expression can be bounded by the No Bonds Theorem. Delete from X_1 a unit disk around both points of each bond belonging to $\mathcal{F}$, and restrict the Poisson process to the remaining set. The probability that $\mathcal{F}$ is maximal is equal to the probability that no bonds are formed under the restricted process. Hence, this probability is at most $e^{-\mu'+\nu'}$, where $\mu' < \mu$ and $\nu' < \nu$ are the expected number of bonds and "vees" in the restricted process, respectively. If $k > (4/7)\log_2 n$, then $\mu' > \mu/2 > 2\nu$. This yields that the conditional probability in (11) is at most $e^{-\mu'+\nu} < e^{-\mu/2+\nu} < e^{-\mu/4}$. Therefore, (11) can be bounded from above by

$$\sum_{i<\frac{\mu}{10}} \frac{\mu^i}{i!} e^{-\mu/4} < e^{-\mu/8},$$

as required. □

Lemma 6.2 *The probability that there are at least $\mu/10$ pairwise separated bad bonds is at most $e^{-\mu/8}$.*

Proof: The expected number of *bad* bonds is at most $2\varepsilon\pi\mu$, because for any bond, the probability that there is a point in the unit disk centered at one of its points is $\varepsilon\pi$. If two point pairs are separated, then the events that they form bad bonds are *independent.* Thus, the probability that there exist $\mu/10$ bad bonds is at most

$$\frac{(2\varepsilon\pi\mu)^{\mu/10}}{(\mu/10)!} < (100\varepsilon)^{\mu/10} < e^{-\mu/8}.$$

□

It follows from Lemmas 6.1 and 6.2 that if j is *moderate* and $k \geq (4/7)\log_2 n$, then the probability that P_1^* contains two points whose distance belongs to I_{jkl} satisfies

$$\Pr{}^*[I_{jkl} \text{ is empty}] < 2e^{-\mu/8}.$$

Thus, using the same estimates as in 5.1, we obtain

$$\sum_{j \text{ moderate}} \sum_{k \geq \frac{4}{7}\log n} \sum_{l=1}^{2^k} |I_{jkl}|^2 \Pr{}^*[I_{jkl} \text{ is empty}] = O(n^{-6/7}). \tag{12}$$

On the other hand, the probability that there exist a moderate j, $k \leq (4/7)\log_2 n$, and l such that the distance of no two points of P_1^* belongs to the canonical interval I_{jkl} is at most

$$O\left(\sum_{j=1}^{1.96n^{4/7}} 2^{-(4/7)\log n} \Pr{}^*[I_{j\lceil(4/7)\log n\rceil l} \text{ is empty}]\right) = \exp(-\Omega(n^{3/7})).$$

The above argument can also be applied to *large* distances. Then,

$$\sum_{j \text{ large}} \sum_{k \geq \frac{4}{7}\log n} \sum_{l=1}^{2^k} |I_{jkl}|^2 \Pr{}^*[I_{jkl} \text{ is empty}] = O(n^{-6/7}), \tag{13}$$

and the probability that there exist a *large* j, $k \leq (4/7)\log_2 n$, and l such that the distance of no two points of P_2^* belongs to the canonical interval I_{jkl} is at most $\exp(-\Omega(n^{2/7}))$.

Summarizing: With probability $1 - o(1)$, $|P_1^* \cup P_2^* \cup P_3| \leq |P_1 \cup P_2 \cup P_3| = O(n)$. With probability at least $1 - \exp(-\Omega(n^{2/7}))$, for every canonical interval I_{jkl} for which j is moderate or large and $k \leq (4/7)\log_2 n$, there is a pair of points in $P_1^* \cup P_2^* \cup P_3$, whose distance belongs to I_{jkl}. By (10), (12), and (13), the expected value of the sum of the squares of all other empty canonical intervals satisfies

$$E\,[\sum_{\substack{I_{jkl} \text{ empty} \\ j \text{ moderate or large} \\ k \leq (4/7)\log n}} |I_{jkl}|^2\,] + \sum_{\substack{I_{jkl} \text{ empty} \\ j \text{ extra large}}} |I_{jkl}|^2 = O(n^{-6/7}).$$

Hence, by Markov's inequality, with probability at least 1/2, the sum of the squares of *all* canonical intervals will be $O(n^{-6/7})$. In view of (7), this implies that there exists a *specific* $O(n)$-element point set $P_1^* \cup P_2^* \cup P_3$ with minimum distance one, for which the sum of the squares of the distance gaps is $O(n^{-6/7})$. This completes the proof of Theorem 1.

References

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