Coloring axis-parallel rectangles

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Abstract

For every k and r, we construct a finite family of axis-parallel rectangles in the plane such that no matter how we color them with k colors, there exists a point covered by precisely r members of the family, all of which have the same color. For $r = 2$, this answers a question of S. Smorodinsky [S06].

1 Introduction

Given a set of points P and a family of regions $\mathcal{R}$ in the plane, in a natural way one can associate two hypergraphs with them, dual to each other. Let $H(P, \mathcal{R})$ denote the hypergraph on the vertex set P, whose hyperedges are all subsets of P that can be obtained by intersecting P with a member of $\mathcal{R}$. The hypergraph $H^*(P, \mathcal{R})$ is defined by swapping the roles of $\mathcal{R}$ and P: its vertex set is $\mathcal{R}$, and for each $p \in P$ it has a hyperedge consisting of all regions in $\mathcal{R}$ that contain p.

Let H be a hypergraph with vertex set $V(H)$. The *chromatic number* $\chi(H)$ of H is the smallest number of colors in a coloring of $V(H)$ such that no hyperedge with at least two vertices is monochromatic. Let H_r (and $H_{\geq r}$) denote the hypergraph on the vertex set $V(H)$, consisting of all r-element (at least r-element) hyperedges of H. By definition, we have $\chi(H) = \chi(H_{\geq 2})$.

In the special case when $\mathcal{R}$ is the family of all *axis-parallel rectangles* and P is a set of n points in the plane, the problem of bounding $\chi(H(P, \mathcal{R}))$ reduces to estimating the chromatic number of the graph $G(P, \mathcal{R}) := H_2(P, \mathcal{R})$ consisting of all *two-element* (hyper)edges of $H(P, \mathcal{R})$. Kříž and Nešetřil

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[KN91] gave an explicit construction showing that the chromatic number of $G(P, \mathcal{R})$ cannot be bounded by an absolute constant. Chen, Pach, Szegedy, and Tardos [CPST07] proved by a probabilistic argument that there exist n-element point sets P such that the chromatic numbers of $G(P, \mathcal{R})$ grow as fast as at least $\Omega\left(\frac{\log n}{\log^2 \log n}\right)$. On the other hand, Ajwani, Elbassioni, Govindarajan, and Ray [AEGR07] proved that $\chi(H(P, \mathcal{R})) = \chi(G(P, \mathcal{R})) \leq O\left(n^{.383}\right)$.

It was also shown in [CPST07] that, for a fixed $r \geq 2$, a randomly and uniformly selected set P of n points in the unit square almost surely satisfies

$$\chi(H_r(P, \mathcal{R}) = \Omega\left(\frac{\log^{1/(r-1)} n}{\log^2 \log n}\right),$$

as n tends to infinity.

Concerning the dual question, S. Smorodinsky [S06] proved that if $\mathcal{R}$ is a family of n open axis-parallel rectangles and P is a set of points in the plane, then $\chi(H^*(P, \mathcal{R})) = O(\log n)$. In other words, the rectangles in $\mathcal{R}$ can be colored by at most constant times $\log n$ colors so that, for any point $p \in P$ covered by more than one rectangle, at least two rectangles containing p have different colors. Smorodinsky asked whether there always exists such a coloring with a bounded number of colors. In the present note, we answer this question in the negative, in the following stronger form

Theorem 1 *Let $n \geq r \geq 2$ be integers. There exists a family $\mathcal{R}$ of n axis-parallel rectangles in the plane such that for any coloring of the rectangles with $k \leq C\left(\frac{\log n}{r \log r}\right)$ colors, one can find a point covered by exactly r members of $\mathcal{R}$, all of the same color. Here $C > 0$ is an absolute constant.*

Using our notation, we have that $\chi(H_r^*(P, \mathcal{R})) \geq C\left(\frac{\log n}{r \log r}\right)$, where $P = \mathbb{R}^2$ (or a suitable finite subset of $\mathbb{R}^2$). For $r = 2$, the above mentioned result of Smorodinsky [S06] shows that Theorem 1 is not far from being optimal.

A family of axis-parallel rectangles is said to form a *r-fold covering* of the plane if every point $p \in \mathbb{R}^2$ is contained in at least r members of the family. It is called *locally finite* if no point of the plane belongs to infinitely many rectangles. We say that a covering has a *k-split* if the family can be partitioned into k parts such that the union of any $k-1$ parts form a (1-fold) covering. Theorem 1 yields

Corollary 2 *For every $r, k \geq 2$, there is a locally finite r-fold covering of the plane with axis-parallel rectangles that does not have a k-split.*

In the case $k = 2$, Corollary 2 states that there are locally finite r-fold coverings of the plane with axis-parallel rectangles that cannot be partitioned into two coverings. A very simple direct construction proving this can be found in [PTT07]. Some positive results with half-planes, disks, translates of a convex polygon, etc., in the place of rectangles, were established in [P80], [MP87], [P86], [TT06], [K07], and [ACCLS07]. There is an intimate relationship between questions of this type and the notion of *conflict-free* colorings, introduced by Even, Lotker, Ron, and Smorodinsky [ELRS03]; see also [HS05]. Many similar problems on colorings are discussed in [BMP05] and [MP06].

2 The construction and its basic properties

First we have to introduce some notations.

For any two integers $c \geq 2$ and $k \geq 0$, let $[c] := \{0, 1, \dots, c-1\}$ and let $[c]^k$ stand for the set of strings of length k over the alphabet $[c]$. For $x \in [c]^k$, let x_j denote the jth digit of x ($1 \leq j \leq k$), so that we have $x = x_1 \dots x_k$. Let $\overleftarrow{x}$ denote the reverse of x, that is, $\overleftarrow{x} = x_k \dots x_1$. An initial segment of x is a string $x_1 \dots x_j$ for some $0 \leq j \leq k$. Expanding x as a c-ary fraction, we obtain a number $\overline{x} := \sum_{j=1}^{k} x_j / c^j$.

Let $c \geq 2$ and $d \geq 1$ be integers. For any $0 \leq k \leq d$, $u \in [c]^k$, and $v \in [c]^{d-k}$, define an open axis-parallel rectangle $R^k_{u,v}$ in the plane as follows:

$$R^k_{u,v} := (\overline{u}, \overline{u} + c^{-k}) \times (\overline{v}, \overline{v} + c^{k-d}).$$

Now we are in a position to define the family of rectangles $\mathcal{R}$ meeting the requirements of Theorem 1. Let $\mathcal{R}$ consist of the rectangles in

$$\left\{ R^k_{u,v} \;\mid\; 0 < k < d, u \in [c]^k, v \in [c]^{d-k}, u_k = v_{d-k} \right\}$$

together with the rectangles in

$$\left\{ R^0_{\varepsilon,v} \;\mid v \in [c]^d, v_d = 0 \right\} \cup \left\{ R^d_{u,\varepsilon} \;\mid\; u \in [c]^d, u_d = 0 \right\},$$

where ε stands for the empty string. For convenience, we slightly change the notation. For any $z \in [c]^{d-1}$, $0 < k < d$, set $S^k_z := R^k_{u,v}$, where u is the initial segment of z of length k, and v is the initial segment of $\overleftarrow{z}$ of length $d - k$. Further, for any $z \in [c]^{d-1}$, set $S^0_z := R^0_{\varepsilon,v}$ and $S^d_z := R^d_{u,\varepsilon}$, where u

is obtained from z by appending to it a 0 as its last digit, and v is obtained from $\overleftarrow{z}$ in the same way. Using this notation, we have

$$\mathcal{R} = \mathcal{R}(c,d) = \{S_z^k \mid 0 \leq k \leq d, z \in [c]^{d-1}\}.$$

Clearly, we have $|\mathcal{R}| = (d+1)c^{d-1}$. Finally, let $H^* = H^*(c,d) = H^*(\mathbb{R}^2, \mathcal{R}(c,d))$ denote the hypergraph on the vertex set $\mathcal{R} = \mathcal{R}(c,d)$, whose hyperedges are all nonempty subsets of $\mathcal{S} \subseteq \mathcal{R}$ for which there is a point in the plane covered by the elements of $\mathcal{S}$, but by no other element of $\mathcal{R}$.

The most important property of our construction is the following.

Theorem 3 *Let $d > 0$, $2 \leq r < c$, and let $H^* = H^*(c,d)$ denote the hypergraph defined above. If a subset $I \subseteq \mathcal{R}(c,d)$ contains no hyperedge of H^* of size r, then we have*

$$|I| \leq \frac{c^{d-1}}{\frac{1}{r-1} - \frac{1}{c-1}}.$$

Let $G^* := G^*(c,d)$ denote the graph H_2^*, consisting of all two-element hyperedges of $H^* = H^*(c,d)$. In view of Theorem 3, the chromatic number $\chi(H^*(c,d))$ satisfies the following.

Corollary 4 *(1) The graph $G^*(2,d)$ is bipartite.*
(2) The chromatic number of $G^*(3,d)$ is at least $\frac{d+1}{2}$.
(3) For $c \geq d+3$, the chromatic number of $G^*(c,d)$ is $d+1$.

Proof: To establish (1), color the rectangles $S_z^k \in \mathcal{R}(2,d)$ according to the parity of $k + \sum_{i=1}^{d} z_i$. Clearly, this is a proper coloring of the vertex set of $G^*(2,d)$.

Applying Theorem 3 with $r = 2$, we obtain that the size of every independent set in $G^*(c,d)$ is at most $c^{d-1}/(1-1/(c-1))$. As the total number of vertices of $G^*(c,d)$ is $(d+1)c^{d-1}$, the chromatic number of $G^*(c,d)$ is at least $(d+1)(1-1/(c-1))$. In case $c = 3$, this gives the bound claimed in (2). For $c \geq d+3$, we have $(d+1)(1-1/(c-1)) > d$, so that $\chi(G^*(c,d)) \geq d+1$. This is tight, since the vertices $S_z^k \in V(G^*(c,d)) = \mathcal{R}(c,d)$ can be colored according to the value k, $0 \leq k \leq d$. □

Theorem 1 immediately follows from

Corollary 5 *Let $k, r \geq 2$ be fixed. There exists a family of $k(2r)^{2kr}$ axis-parallel rectangles in the plane such that for any coloring of these rectangles with k colors, one can find a point covered by exactly r rectangles, all of which have the same color.*

Proof: Consider the family $\mathcal{R}(c,d)$ with $c = 2r$, $d = 2kr - 1$. For any k-coloring of the members of $\mathcal{R}(c,d) = \mathcal{R}(2r, 2kr-1)$, the size of the largest color class is at least $(d+1)c^{d-1}/k = c^d$, which is larger than the bound in Theorem 3. Thus, the largest color class contains a hyperedge of $H^*(c,d)$ of size r. Note that the slightly smaller choices $c = 2r-1$, $d = 2kr-2k$ would also suffice for the proof. $\square$

It remains to establish Theorem 3.

3 Proof of Theorem 3

Let $d \geq 1$ and $c \geq 2$ be fixed. Let $\overline{\mathcal{R}} = \overline{\mathcal{R}}(c,d)$ denote the family of all rectangles $R^k_{u,v}$ with $0 \leq k \leq d$, $u \in [c]^k$, $v \in [c]^{d-k}$. Let $\overline{H}^* = \overline{H}^*(c,d)$ be the corresponding hypergraph, that is, let the vertices of $\overline{H}^*$ be the members of $\overline{\mathcal{R}}$, and let its hyperedges be all sets of the form $\{R \in \overline{\mathcal{R}} \mid p \in R\}$, where $p \in \mathbb{R}^2$.

First we study the structure of $\overline{H}^*$. We define $R^k_{u,v} \leq R^l_{w,z}$ if $k \leq l$, u is an initial segment of w, and z is an initial segment of v. Note that $\leq$ is a partial order on $\overline{\mathcal{R}}$. We show that the hyperedges of $\overline{H}^*$ form intervals in this partial order.

Lemma 6 *The hyperedges of $\overline{H}^*$ are exactly the sets $\{e \in \overline{\mathcal{R}} \mid a \leq e \leq b\}$, where $a \leq b$ are elements of $\overline{\mathcal{R}}$.*

Proof: Let $a = R^k_{u,v}$ and $b = R^l_{w,z}$ be two elements of $\overline{\mathcal{R}}$ with $a \leq b$. Let $p_{a,b} = (\overline{w} + c^{-l-1}, \overline{v} + c^{k-d-1})$, a point of the plane. Let $e = R^m_{x,y} \in \overline{\mathcal{R}}$ arbitrary. We have $e = (\overline{x}, \overline{x} + c^{-m}) \times (\overline{y}, \overline{y} + c^{m-d})$. So $p_{a,b} \in e$ if and only if $\overline{x} < \overline{w} + c^{-l-1} < \overline{x} + c^{-m}$ and $\overline{y} < \overline{v} + c^{k-d-1} < \overline{y} + c^{m-d}$. The first pair of inequalities is satisfied if and only if $m \leq l$ and x is an initial segment of w, while the second pair is satisfied if and only if $k \leq m$ and y is an initial segment of v. Both pairs are satisfied if and only if $a \leq e \leq b$. Therefore, the point $p_{a,b}$ shows that $\{e \in \overline{\mathcal{R}} \mid a \leq e \leq b\}$ is an edge of $\overline{H}^*$.

To see that $\overline{H}^*$ has no additional hyperedges, it is sufficient to prove the following two claims.

(1) Any two rectangles $a, b \in \overline{\mathcal{R}}$ are disjoint, unless $a \leq b$ or $b \leq a$.

(2) For any three rectangles $a \leq e \leq b$, we have $a \cap b \subseteq e$.

Here (1) shows that the vertices of any hyperedge of $\overline{H}^*$ is linearly ordered by $\leq$. If a is the minimal vertex of a hyperedge f and b is its maximal vertex, then $f \subseteq \{e \in \overline{\mathcal{R}} \mid a \leq e \leq b\}$, and by (2) equality must hold.

To verify (1), let $a = R^k_{u,v} = a_1 \times a_2$ and $b = R^l_{w,z} = b_1 \times b_2$. Here a_1, a_2, b_1, and b_2 are open intervals. We may assume $k \leq l$, by symmetry. If for some $i \leq k$ we have $u_i \neq w_i$, then a_1 and b_1 are disjoint. If for some $i \leq d-l$, we have $v_i \neq z_i$, then a_2 and b_2 are disjoint. So if a and b intersect, we must have $a \leq b$. Notice that in this case we have $b_1 \subseteq a_1$ and $a_2 \subseteq b_2$.

To verify (2), let $a \leq e \leq b$ with $a = a_1 \times a_2$ and $b = b_1 \times b_2$. Using the last observation of the previous paragraph, we have $a \cap b = b_1 \times a_2$ and by the same observation again, this is contained in e. $\square$

The *type* of a rectangle $R^k_{u,v} \in \overline{\mathcal{R}}$ is k. It is easy to see that if $a \leq b$ are rectangles in $\overline{\mathcal{R}}$ of type k and l, respectively, then for all $k \leq m \leq l$, there exists precisely one $d \in \overline{\mathcal{R}}$ of type m that satisfies $a \leq d \leq b$. Consider now the two-coloring of $\overline{\mathcal{R}}$, where the color of a rectangle is determined by the parity of its type. The last observation shows that in any hyperedge of $\overline{H}^*$ the number of vertices of the two color classes differ by at most one. In particular, no edge of size at least two is monochromatic.

Next we describe the structure of the subfamily $\mathcal{R} = \mathcal{R}(c,d)$ of $\overline{\mathcal{R}}$. The elements of $\mathcal{R}$ are partially ordered by $\leq$. Note that for any $a = S^k_z$ and $b = S^l_t$, we have $a \leq b$ if and only if $k \leq l$ and $z_i = t_i$ for all $1 \leq i \leq k$ and $l \leq i \leq d-1$. For any $a \leq b$ in $\mathcal{R}(c,d)$, we define the *interval* $[a,b] = \{e \in \mathcal{R} \mid a \leq e \leq b\}$. For any $a = S^k_z$ and $b = S^l_t$ with $a \leq b$, the interval $[a,b]$ contains one element of type m for each index m such that $k \leq m \leq l$ and $z_m = t_m$. Here the type of the rectangle S^m_w (inherited from $\overline{\mathcal{R}}$) is m.

Corollary 7 *The hyperedges of $H^*(c,d)$ are exactly the intervals $[a,b]$, where $a \leq b$ are vertices of $H^*(c,d)$.*

Proof: The hyperedges of $H^*(c,d)$ are the sets $e \cap \overline{\mathcal{R}}$ where e is a hyperedge of $\overline{H}^*(c,d)$. The assertion follows from Lemma 6. $\square$

Corollary 8 *Two vertices S^k_z and S^l_t are connected in $G^*(c,d)$ by an edge if and only if*

(1) $k \neq l$,

(2) for all indices i strictly between k and l, we have $z_i \neq t_i$, and

(3) for all other indices i, we have $z_i = t_i$. $\square$

Proof of Theorem 3: Let us fix $d \geq 1$, $2 \leq r < c$ and a set $I \subseteq \mathcal{R} = \mathcal{R}(c,d)$ such that no edge of $H^*(c,d)$ of size r is contained in I. For any vertex $a \in \mathcal{R}$ of type $0 \leq i < d$, we define the *next* vertex $N(a)$ of type $i+1$ as follows. Let $N(S^i_x) = S^{i+1}_x$ if $i = 0$ or $S^i_x \in I$. If $0 < i < d$ and $S^i_x \notin I$, then let $N(S^i_x) = S^{i+1}_y$, where $y_j = x_j$ for all indices $1 \leq j \leq d-1$, $j \neq i$, and $y_i = (x_i + s) \bmod c$. We choose s to be the smallest positive integer such that $S_y^i \notin I$. In other words, we obtain y from x by shifting the ith digit cyclically upward until we reach a value y with $S_y^i \notin I$. Note that the choice $s = c$ makes $y = x$ and $S_y^i \notin I$. We call the vertex S_x^i *bad* if $s = c$ is the minimal choice, that is, if $1 \leq i \leq d-1$, $S_x^i \notin I$ and $N(S_x^i) = S_x^{i+1}$. Clearly, S_x^i is bad if and only if $S_x^i \notin I$ but $S_y^i \in I$ for all the $c-1$ strings y that differ from x only at the position i. Therefore, for the set B of bad vertices, we have $|B| \leq |I|/(c-1)$.

It is easy to see that for $0 \leq i \leq d-1$, the function $N : \mathcal{R}(c,d) \rightarrow \mathcal{R}(c,d)$ is a one-to-one mapping from the vertices of type i to the vertices of type $i+1$. We call a sequence $a_0, a_1, \ldots, a_d$ of vertices of $\mathcal{R}$ a *cluster* if $N(a_i) = a_{i+1}$ for $0 \leq i \leq d-1$. Note that a_i is of type i for every i. There are c^{d-1} clusters (one for each a_0 of type 0) and they form a partition of $\mathcal{R}$ into subsets of size $d+1$.

The main observation is the following. If in a cluster $a_0, a_1, \ldots, a_d$ we have $a_i, a_j \in I$ for some indices $0 \leq i \leq j \leq d$, then $a_i \leq a_j$ and $[a_i, a_j] = \{a_k \mid i \leq k \leq j,\, a_k \in I \cup B\}$. This follows readily from the definition of the relation $\leq$.

Now we use that intervals like $[a_i, a_j]$ are edges of the hypergraph $H^*(c,d)$ (Corollary 7), and no edge of size r is contained in I. Therefore, a cluster contains at most $r-1$ elements of I not separated by bad vertices. If there are b bad vertices in a cluster, then the cluster contains at most $(r-1)(b+1)$ elements of I. Summing over all clusters, we conclude that

$$|I| \leq (r-1)|B| + (r-1)c^{d-1}.$$

Using the inequality $|B| \leq |I|/(c-1)$ and rearranging the terms, the desired bound on $|I|$ follows. □

4 Concluding remarks

For any $c \geq 2, d \geq 1$, define a graph $G' = G'(c,d)$ on the vertex set $\{v_z^k \mid 0 \leq k \leq d, z \in [c]^d\}$ as follows. Connect two vertices v_z^k and v_t^l with $k \leq l$ by an edge if and only if

(1) $k < l$,

(2) for all indices i with $k < i \leq l$, we have $z_i \neq t_i$, and

(3) for all other indices i, we have $z_i = t_i$.

Clearly, the definition of $G'(c,d)$ is very close to the description of $G^*(c,d)$ given in Corollary 8. Let us compare the two graphs.

It is easy to see that G' contains a complete graph on $d+1$ vertices. For example, let $z(i)$ be a string of i zeros followed by $d-i$ ones. Notice that the vertices $v^i_{z(i)}$, $0 \le i \le d$, form a complete subgraph. Thus, the chromatic number of G' is at least $d+1$, and this is tight, as shown by the coloring that assigns color i to all vertices of the form v^i_z. Moreover, the vertex set of G' can be partitioned into $(d+1)$-element sets so that each set induces a complete graph in G'. To see this, let $x = x_1 \dots x_d \in [c]^d$, and for any $0 \le i \le d$, define $x(i) \in [c]^d$ as follows: let $(x(i))_j = x_j$ if $j > i$, and let $(x(i))_j = (x_j + 1) \bmod c$ if $j \le i$. Clearly, the vertices $v^i_{x(i)}$ induce a complete subgraph in G', and for the different choices of x, they form a vertex partition. This shows that no independent set in G' contains more than a fraction of $1/(d+1)$ of the vertices.

On the other hand, the graph G^* is triangle-free. To verify this, consider three vertices $a = S^k_z$, $b = S^l_t$, and $e = S^m_w$ with $k \le l \le m$. If a and b are connected by an edge in G^*, we have $k < l$ and $z_l = t_l$. Analogously, if b and e are connected, then $l < m$ and $t_l = w_l$. This yields that $k < l < m$ and $z_l = w_l$, which implies that a and e are not connected. Despite this difference, the proof of Theorem 3 was inspired by the similarity between G^* and G', and by the simple argument above, which provides an upper bound on the size of the largest independent set in G'.

The fact that G^* is triangle-free is not merely a coincidence. Consider any family $\mathcal{S}$ of axis-parallel rectangles in the plane, and construct a graph $G = (V, E)$ on the vertex set $V = \mathcal{S}$ by connecting two rectangles if they have a point in common that is not covered by any other member of $\mathcal{S}$. Recall that G^* was also constructed in this manner. While G may contain triangles and even subgraphs isomorphic to K_4, it cannot have a complete subgraph on *five* vertices. Furthermore, we can partition the edge set of G into two subsets $E = E_1 \cup E_2$ as follows. Let us put an edge $\{v_1, v_2\} \in E$ in E_1 if the boundaries of the rectangles v_1 and v_2 cross in *four* points. Otherwise, put the edge $\{v_1, v_2\}$ in E_2. It is easy to see that $G_1 := (V, E_1)$ is triangle-free. It seems that G_1 is the interesting part of G, as the chromatic number of the graph $G_2 := (V, E_2)$ is small. Perhaps G_2 is even Δ-*degenerate* for an appropriate absolute constant Δ, that is, every subgraph of G_2 has a vertex of degree at most Δ. To prove this, it would be sufficient to show that $|E_2| \le \Delta |V|/2$.

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