Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles

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Abstract

To solve frequency assignment problems in cellular telephone networks, Even, Lotker, Ron, and Smorodinsky (FOCS 02) introduced the notion of conflict-free colorings in various geometrically defined hypergraphs. They initiated the investigation of the special case when the vertex set of the hypergraph is a set P of n points in the plane, and the hyperedges are those subsets of P that can be obtained by intersecting P with an axis-parallel rectangle. The 2-element subsets of P satisfying this condition form (the edge set of) the *Delaunay graph* $D(P)$ associated with P. The problem of estimating the minimum number of colors in a conflict-free coloring leads to the following question: Does there exist a constant $c > 0$ such that the Delaunay graph of any set of n points in the plane contains an independent set of size at least cn? We answer this question in the negative. We also show that for a set P of n randomly and uniformly selected points in the unit square, $D(P)$ has an independent set of size at least $cn/\log n$, with probability tending to 1. We generalize these results to solve a problem in geometric discrepancy theory.

1 Delaunay graphs and conflict-free colorings

The *Delaunay graph* associated with a set of points P in the plane is a graph $D(P)$ whose vertex set is P and whose edge set consists of those pairs $\{p, q\} \subset P$ for which there exists a closed disk that contains p and q, but does not contain any other element of P. The Delaunay graph of P is a planar graph and its dual is

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the *Dirichlet–Voronoi diagram* of P (see, e.g., [BKOS00]). As any other planar graph, $D(P)$ contains an independent set of size at least $|P|/4$. It was discovered by Even, Lotker, Ron, and Smorodinsky [ELRS03] that this fact easily implies that any set P of n points in the plane has a *conflict-free coloring* with respect to discs, which uses at most $O(\log n)$ colors, that is, a coloring with the property that any closed disk C with $C \cap P \neq \emptyset$ has an element whose color is not assigned to any other element of $C \cap P$. Here, the logarithmic bound is tight for every point set [PaT03].

The question was motivated by a frequency assignment problem in cellular telephone networks. The points correspond to *base stations* interconnected by a fixed backbone network. Each *client* continuously scans frequencies in search of a base station within its (circular) range with good reception. Once such a base station is found, the client establishes a radio link with it, using a frequency not shared by any other station within its range. Therefore, a conflict-free coloring of the points corresponds to an assignment of frequencies to the base stations, which enables every client to connect to a base station without interfering with the others. For many results on conflict-free colorings, consult [AlS06], [FiLM05], [HaS05].

The same scheme can be used to construct conflict-free colorings of point sets with respect to various other families of geometric figures. In general, let P be a set of points in $\mathbf{R}^d$, and let $\mathcal{C}$ be a family of d-dimensional convex bodies. Define the *Delaunay graph* $D_{\mathcal{C}}(P)$ *of* P *with respect to* $\mathcal{C}$ on the vertex set P by connecting two elements $p, q \in P$ with an edge if and only if there is a member of $\mathcal{C}$ that contains p and q, but no other element of P. The existence of large independent sets in such graphs implies that P has a *conflict-free coloring with respect to* $\mathcal{C}$, which uses a small number of colors. That is, a coloring with the property that any member $C \in \mathcal{C}$ with $C \cap P \neq \emptyset$ has an element whose color is not assigned to any other element of $C \cap P$.

In this note, we consider this problem in the special case when $\mathcal{C}$ is the family of *axis-parallel boxes*. The maximum size of an independent set of vertices in a graph G is called the *independence number* of G, and is usually denoted by $\alpha(G)$ in the literature. Smorodinsky et al. [ELRS03], [HaS05] asked whether the Delaunay graph of every set of n points in the plane with respect to axis-parallel rectangles has independence number at least cn, for an absolute constant $c > 0$. In Section 3, we give a negative answer to this question. More precisely, we establish

Theorem 1. *There are n-element point sets in the plane such that the independence numbers of their Delaunay graphs with respect to axis-parallel rectangles are at most* $O\left(n\frac{\log^2 \log n}{\log n}\right)$.

In fact, a randomly and uniformly selected set of n points in the unit square will meet the requirements with probability tending to 1.

For randomly selected point sets, this result is not far from being best possible. In Section 2, we prove

Theorem 2. *The expected value of the independence number of a randomly and uniformly selected n-element point set in the units square is $\Omega\left(\frac{n}{\log n}\right)$.*

For general point sets, we know only a very weak bound: the independence number of the Delaunay graph of any set of n points in the plane with respect to axis-parallel rectangles is at least $\Omega\left(\sqrt{n \log n}\right)$. This only implies that any set of n points in the plane admits a conflict-free coloring using $O(\sqrt{n/\log n})$ colors, with respect to the family of all axis-parallel rectangles.

In discrepancy theory [BeCh87], [Ch00], [Ma99], there are plenty of results that indicate some unavoidable irregularities in geometric configurations. In Section 4, we generalize Theorem 1. Our results immediately imply

Theorem 3. *For any constants $c, d > 1$, a randomly and uniformly selected set P of n points in the unit square almost surely has the following property. For any coloring of the elements of P with c colors, there always exists an axis-parallel rectangle with at least d points in its interior, all of which have the same color.*

2 Delaunay graphs of random point sets

The aim of this section is to prove Theorem 2.

Let $P = \{(x_i, y_i) : 1 \leq i \leq n\}$ be a point set in the unit square, whose no two elements share the same x-coordinate or y-coordinate. Clearly, the Delaunay graph $D(P)$ with respect to axis-parallel rectangles depends only on the relative position of the points in P and not on their actual coordinates. That is, there exists a permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that for the set $P' = \{(i, \pi(i)) : 1 \leq i \leq n\}$ we have $D(P) = D(P')$. Moreover, for a random set of points in the square, the corresponding permutation π is uniformly random. With a slight abuse of notation, we write $D(\pi)$ for the Delaunay graph $D(P) = D(P')$. In our arguments about Delaunay graphs of randomly selected point sets in the square, it will be convenient to consider the graph $D(\pi)$ for a *random permutation* π.

Lemma 1. *Let $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a random permutation, and let $\overline{deg}(D(\pi))$ denote the average degree of the vertices of the Delaunay graph $D(\pi)$. The expected value of the average degree satisfies*

$$\mathsf{E}(\overline{deg}(D(\pi))) = \Theta(\log n).$$

Proof. Two points $p_i = (i, \pi(i))$ and $p_j = (j, \pi(j))$ with $i < j$ are connected by an edge in $D(P)$ if and only if $\pi(i)$ and $\pi(j)$ are consecutive elements in the natural ordering of the set $S = \{\pi(k) | i \leq k \leq j\}$. Among all $\binom{j-i+1}{2}$ pairs of elements in this set, precisely $j - i$ consist of consecutive elements. Clearly, after fixing $\pi(k)$ for $k < i$ or $k > j$, the pair $\{\pi(i), \pi(j)\}$ is equally likely to be any one of the pairs in S. Therefore, the probability that p_i and p_j are connected is equal to

$$\frac{j-i}{\binom{j-i+1}{2}} = \frac{2}{j-i+1}.$$

Thus, the expected number of edges in $D(P)$ is

$$\sum_{l=1}^{n-1} \frac{2(n-l)}{l+1} = (2n+2) \sum_{l=1}^{n} \frac{1}{l} - 4n = \Theta(n \log n).$$

$\square$

Obviously, Theorem 2 is equivalent to

Theorem 2'. *Let π be a random permutation of $\{1, 2, \ldots, n\}$. The expected value of the independence number of the Delaunay graph $D(\pi)$ with respect of axis-parallel rectangles satisfies*

$$\mathsf{E}[\alpha(D(\pi))] = \Omega\left(\frac{n}{\log n}\right).$$

Proof. According to Turán's theorem, any graph with n vertices and average degree d has an independent set of size at least $\frac{n}{d+1}$. Thus, we have

$$\alpha(D(\pi)) \geq \frac{n}{\overline{\deg}(D(\pi)) + 1}.$$

By the convexity of the $x \to n/(x+1)$ function for $x \geq 0$, we have

$$\mathsf{E}[\alpha(D(\pi))] \geq \frac{n}{\mathsf{E}[\overline{\deg}(D(\pi)) + 1}$$

and the theorem follows by Lemma 1. $\square$

3 Proof of Theorem 1

We reformulate and prove Theorem 1 in a more precise form.

Theorem 1'. *Let P be a set of n randomly and uniformly selected points in the square $[0,1]^2$. Then there exists a constant c such that*

$$\mathrm{Prob}_{n\to\infty}\left(\alpha(D(P)) < c\frac{n\log^2\log n}{\log n}\right) \to 1.$$

Proof. The points $p_i \in P$ will be defined in two steps. First we select the x-coordinates from the interval $[0,1]$ uniformly at random. With probability 1, all the x coordinates are distinct. Let us relabel the points so that

$$0 \le x_1 < x_2 < \cdots < x_n \le 1.$$

In the second step, we select the y-coordinates of $p_i = (x_i, y_i)$ uniformly and independently from $[0,1]$. Note that, after the x_i's have been fixed, the edge set of the Delaunay graph $D(P)$ depends only on the relative order of the y_i's.

The coordinates y_i are generated as follows. Fix an integer $L \ge 2$ to be specified later. We write the numbers $y_i \in [0,1]$ in base L:

$$y_i = (0.d_i^{(1)}d_i^{(2)}\dots)_L.$$

The digits $d_i^{(t)}$ of y_i are chosen independently and uniformly from the set $\{0,\dots,L-1\}$. For $t \ge 1$, denote by $y_i^{(t)}$ the truncated L-ary fraction of y_i, consisting of $t-1$ digits after 0:

$$y_i^{(t)} = (0.d_i^{(1)}\dots d_i^{(t-1)})_L.$$

The digits of y_i will be chosen one by one. At *stage* t, we determine $d_i^{(t)}$ (and, hence, $y_i^{(t+1)}$), for all i. Note that *before* stage t, the truncated fractions $y_i^{(t)}$ have already been fixed. As soon as we complete stage t, we know the y-coordinates of the points p_i up to an error of at most L^{-t}. If $y_i^{(t+1)} = y_j^{(t+1)}$, then the relative order of y_i and y_j has not yet been decided. Otherwise, if we have $y_i^{(t+1)} < y_j^{(t+1)}$, say, then $y_i < y_j$ holds in the final configuration.

Let $1 \le i < j \le n$ be fixed. Suppose that for some t, the following two conditions are satisfied:

1. $y_i^{(t+1)} = y_j^{(t+1)}$,

2. $y_k^{(t+1)} \neq y_i^{(t+1)}$ holds for all k satisfying $i < k < j$.

Then the rectangle $[x_i, x_j] \times [y_i^{(t+1)}, y_i^{(t+1)} + L^{-t})$ contains p_i and p_j, but no other element of P. Thus, in this case, p_i and p_j are connected in $D(P)$, and we say that this edge is *forced at stage* t. Although $D(P)$ may contain many edges that are not forced at any stage, we are going to use only forced edges in proving our upper bound on the independence number of $D(P)$.

Let us fix a subset $I \subset \{1, \dots, n\}$, and let $Q = Q(I) = \{p_i : i \in I\}$. We want to estimate from above the probability that Q is an *independent* set in $D(P)$.

Let $t \geq 1$, and consider stage t of our selection process. Before this stage, $y_i^{(t)}$ has been fixed for every i. For any L-ary fraction y of the form $y = (0.d^{(1)}d^{(2)} \cdots d^{(t-1)})_L$, define a subset $H_y \subseteq \{1, \dots, n\}$ by

$$H_y = \{1 \leq i \leq n : y_i^{(t)} = y\}.$$

Obviously, these sets partition $\{1, \dots, n\}$, and hence I, into at most L^{t-1} nonempty parts. If two indices $i, j \in I$ are consecutive elements of the same part $H_y \cap I$, then we call them *neighbors*. That is, $i < j$ are neighbors if

1. $y_i^{(t)} = y_j^{(t)} = y$ holds for some y, and
2. $H_y \cap \{k \in I : i < k < j\} = \emptyset$.

For any two neighbors $i, j \in H_y$ $(i < j)$, define

$$S_{i,j} = \{k \in H_y : i < k < j\}.$$

Two neighbors $i, j \in I$ $(i < j)$ are called *close neighbors* if $|S_{i,j}| \leq L$.

If there are two close neighbors $i, j \in I$ such that the $\{p_i, p_j\}$ is an edge of $D(P)$ forced at stage t, then Q is not an independent set in $D(P)$ and we say that Q *fails at stage* t. Otherwise, Q is said to *survive* stage t, and we indicate this fact by writing $Q \curvearrowright t$.

Let $i < j$ be a pair of close neighbors. Note that $\{p_i, p_j\}$ is an edge of $D(P)$ forced in stage t if and only if $d_i^{(t)} = d_j^{(t)}$, but $d_i^{(t)} \neq d_k^{(t)}$ holds for all $k \in S_{i,j}$. The probability of this event is

$$\mathsf{Prob}(\{p_i, p_j\} \text{ is forced at stage } t) = \frac{1}{L}\left(1 - \frac{1}{L}\right)^{|S_{i,j}|}.$$

Taking into account that $|S_{i,j}| \leq L$, we obtain

$$\mathsf{Prob}(\{p_i, p_j\} \text{ is forced at stage } t) \geq \frac{1}{4L}.$$

Notice that, assuming a fix outcome of previous stages (i.e., $p_k^{(t)}$ is fixed for all k), the presence of edges $\{p_i, p_j\}$ forced at stage t are independent for all neighbors. Thus,

$$\mathsf{Prob}(Q \curvearrowright t | \text{outcome of stages } t' < t) \le \left(1 - \frac{1}{4L}\right)^m \le e^{-\frac{m}{4L}},$$

where m stands for the number of pairs $i, j \in I$ that are close neighbors before stage t.

Obviously, every $i \in I$, except the last element in each set H_y, has exactly *one* neighbor $j > i$. As the sets $S_{i,j}$ are pairwise disjoint for different pairs of neighbors $i < j$, there are fewer than $\frac{n}{L}$ pairs that are neighbors but not close neighbors. Thus, we have

$$m > |I| - \frac{n}{L} - L^{t-1}.$$

If $t \le \log n / \log L$ and $|I| \ge 3n/L$, we have $m \ge n/L$, and thus

$$\mathsf{Prob}(Q \curvearrowright t | \text{outcome of stages } t' < t) \le e^{-\frac{n}{4L^2}}.$$

As the above bound applies assuming any set of choices made at previous stages, so in particular, it applies to the conditional probability that Q survives stage t, given that it has survived all previous stages:

$$\mathsf{Prob}(Q \curvearrowright t | Q \curvearrowright t' \text{ for all } t' < t) \le \left(1 - \frac{1}{4L}\right)^m \le e^{-\frac{n}{4L^2}}.$$

Taking the product of these estimates for all $t \le \log n / \log L$, we obtain

$$\mathsf{Prob}(Q \text{ survives the first } \lfloor \log n / \log L \rfloor \text{ stages}) \le \exp\left(-\frac{n}{4L^2}\left(\frac{\log n}{\log L} - 1\right)\right).$$

The last bound is valid for any set $Q = Q(I) \subseteq P$, where $I \subset \{1, \dots, n\}$ satisfies $|I| \ge 3n/L$. Letting

$$L = \left\lfloor \frac{\log n}{100 \log^2 \log n} \right\rfloor \quad \text{and} \quad a = \left\lceil \frac{3n}{L} \right\rceil,$$

we can conclude that

$$\begin{aligned} \mathsf{Prob}\left(\alpha(D(P)) \ge a\right) &\le \sum_{Q \subset P, |Q| = a} \mathsf{Prob}\left(Q \text{ survives all stages}\right) \\ &\le \binom{n}{a} \exp\left(-\frac{n}{4L^2}\left(\frac{\log n}{\log L} - 1\right)\right) \\ &\to 0, \end{aligned}$$

as required. $\square$

4 Discrepancy in colored random point sets

In this section, we strengthen Theorem 1.

Definition 1. *Given an integer $d > 1$ and a finite point set P in the plane, a subset $Q \subseteq P$ is called* d-independent *if there is no axis-parallel rectangle R such that $|R \cap P| = d$ and $R \cap P \subseteq Q$. Let $\alpha_d(P)$ denote the size of the largest d-independent subset of P.*

According to this definition, a subset of P is 2-independent if and only if it is an independent set in the Delaunay graph $D(P)$ associated with P. In particular, we have $\alpha_2(P) = \alpha(D(P))$.

Obviously, if a set is d-independent for some $d > 1$, then it is also d'-independent for any $d' > d$. Therefore, $\alpha_d(P)$ is increasing in d.

Theorem 3 is a direct corollary to

Theorem 4. *A randomly and uniformly selected set P of n points in the unit square almost surely satisfies*

$$\alpha_d(P) = O\left(\frac{dn \log^2 \log n}{\log^{1/(d-1)} n}\right).$$

Proof. We modify the proof of Theorem 1. Pick the random points $p_i = (x_i, y_i) \in P$ according to the same multi-stage model as in the previous section, and define the truncated fractions $y_i^{(t)}$ that approximate y_i in exactly the same way as before.

Fix a subset $I \subseteq \{1, \dots, n\}$, and let $Q = Q(I) = \{p_i : i \in I\}$. Just like in the proof of Theorem 1, analyze a fixed stage t of the selection process, by introducing the sets H_i.

Instead of using the notion of *neighbors*, we need a new definition. For any two elements $i, j \in I$ $(i < j)$ such that $y_i^{(t)} = y_j^{(t)} = y$ for some y, introduce the sets

$$T_{i,j} = \{k \in H_y \cap I : i \le k \le j\} \quad \text{and} \quad S_{i,j} = \{k \in H_y \setminus I : i < k < j\}.$$

The numbers i and j are called *d-neighbors* if $|T_{i,j}| = d$. The pair $\{i, j\}$ of d-neighbors is called a pair of *close d-neighbors* if $|S_{i,j}| \le L$.

We say that the pair of close d-neighbors $\{p_i, p_j\}$ *fails* at stage t if at this stage the y-coordinates of *all* points p_k with $k \in T_{i,j}$ receive the same new digit $d_k^{(t)} = \delta$, but the y-coordinate of no point p_ℓ with $\ell \in S_{i,j}$ receives this digit. The probability of this event is exactly

$$L^{1-d}\left(1 - \frac{1}{L}\right)^{|S_J|} \ge L^{1-d}\left(1 - \frac{1}{L}\right)^{L} \ge \frac{1}{4L^{d-1}}.$$

Obviously, if any pair $\{p_i, p_j\}$ fails at stage t, then Q cannot be d-independent. In this case, we say that Q *fails at stage* t. Otherwise, Q is said to have *survived* stage t, and we write $Q \curvearrowright t$.

The failures of certain pairs at a given stage are not independent events. However, they are independent for any collection of close d-neighbor pairs (i, j) with the property that the corresponding sets $T_{i,j}$ are pairwise disjoint. To find such a collection consisting of many pairs, select at least $\frac{|H_y \cap I|}{d-1} - 1$ pairs of d-neighbors from each H_y with pairwise disjoint sets $T_{i,j}$, and thus a total of at least $\frac{|I|}{d-1} - L^{t-1}$ pairs. Since the corresponding sets $S_{i,j}$ are pairwise disjoint, all but at most n/L of them are close d-neighbors. Thus, as long as $|I| \geq 3(d-1)n/L$ and $t \leq \log n / \log L$, we obtain collection of

$$m \geq \frac{|I|}{d-1} - L^{t-1} - \frac{n}{L} \geq \frac{n}{L}$$

close d-neighbors with the required property.

If any pair of this collection fails at stage t, then Q fails at this stage. As in the proof of Theorem 1, we have

$$\mathsf{Prob}(Q \curvearrowright t | Q \curvearrowright t' \text{ for all } t' < t) \leq e^{-\frac{n}{4L^d}}$$

and

$$\mathsf{Prob}(Q \text{ survives all stages }) \leq \exp\left(-\frac{n}{4L^d}\left(\frac{\log n}{\log L} - 1\right)\right).$$

Letting

$$L = \left\lfloor \frac{\log^{1/(d-1)} n}{100 \log^2 \log n} \right\rfloor \quad \text{and} \quad a = \left\lceil \frac{3(d-1)n}{L} \right\rceil,$$

we obtain

$$\mathsf{Prob}\left(\alpha(D(P)) \geq a\right) < \binom{n}{a} \exp\left(-\frac{n}{4L^d}\left(\frac{\log n}{\log L} - 1\right)\right) \to 0.$$

□

5 Concluding remarks, open problems

The notion of Delaunay graphs for axis-parallel boxes naturally generalizes to higher dimensions. An easy extension of the proof of Theorem 2 proves that for any fixed d, the Delaunay graph of randomly and uniformly selected points in the

d-dimensional unit cube has expected average degree $O((\log n)^d)$. This implies that random Delaunay graphs have independent sets of size $n^{1-o(1)}$ in higher dimensions, too. All lower bounds that apply to dimension d also apply to every larger dimension. This can easily be seen by projecting a d-dimensional point sets to a coordinate hyperplane. Delaunay graphs can only lose edges under this operation.

In general, by repeated application of the Erdős-Szekeres lemma it is easy to show that the independence number of the Delaunay graph of any set of n points in d-dimensions, with respect to axis-parallel boxes, is at least $\Omega(n^{1/2^{d-1}})$. As far as we know, no significant improvement on this bound is known, although the truth may well be $\Omega(n^{1-o(1)})$, for any fixed d.

Returning to the plane, it is not hard show that the expected number of d-tuples T in a randomly and uniformly selected set P of n points in the plane, for which there exists an axis-parallel rectangle whose intersection with P is T, is $\Theta(d^2 n \log n)$. By a result of Spencer [Sp72], any d-uniform hypergraph with n vertices and $\Theta(nk)$ edges has an independent set of size $\Omega(n/k^{1/(d-1)})$. Therefore, P contains a d-independent subset of size $\Omega(n/\log^{1/(d-1)} n)$. This is within $O(\log^2 \log n)$ of our upper bound.

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