Radial Points in the Plane*

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Abstract

A *radial point* for a finite set P in the plane is a point $q \notin P$ with the property that each line connecting q to a point of P passes through at least one other element of P. We prove a conjecture of Pinchasi, by showing that the number of radial points for a non-collinear n-element set P is $O(n)$. We also present several extensions of this result, generalizing theorems of Beck, Szemerédi and Trotter, and Elekes on the structure of incidences between points and lines.

1 Introduction

Let P be a set of n points in the plane, not all lying on the same line. A point $q \notin P$ is called a *radial point* (for P) if for every line ℓ passing through q we have $|\ell \cap P| \neq 1$. In other words, every line connecting q to some point $p \in P$ passes through at least one other element of P.

For instance, let P be the vertex set of a regular $2k$-gon in the plane. Then, the intersection of the line at infinity with each line supporting an edge of P is a radial point for P. The center of the regular $2k$-gon is another radial point. We thus have a $2k$-element set which has $k+1$ radial points.

Rom Pinchasi [6] conjectured that any non-collinear set of n points in the plane has at most $O(n)$ radial points. He verified this conjecture in the special case when no three points of the set are collinear, and he also established the weaker upper bound $O(n^{3/2})$ for the general case. The main result of our paper is a proof of Pinchasi's conjecture:

Theorem 1. *The maximum possible number of radial points for a non-collinear set of n points in the plane is $\Theta(n)$.*

The construction of Pinchasi depicted in Figure 1 shows that the constant of proportionality in Theorem 1 is at least 5/3 (although it is still possible that the constant is smaller

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if $n > 6$ is sufficiently large). Nevertheless, Pinchasi has shown that the number of radial points of a non-collinear set P of n points in the plane that lie in a halfplane disjoint from P is at most $0.9n$. The constant yielded by our proof is much larger, though.

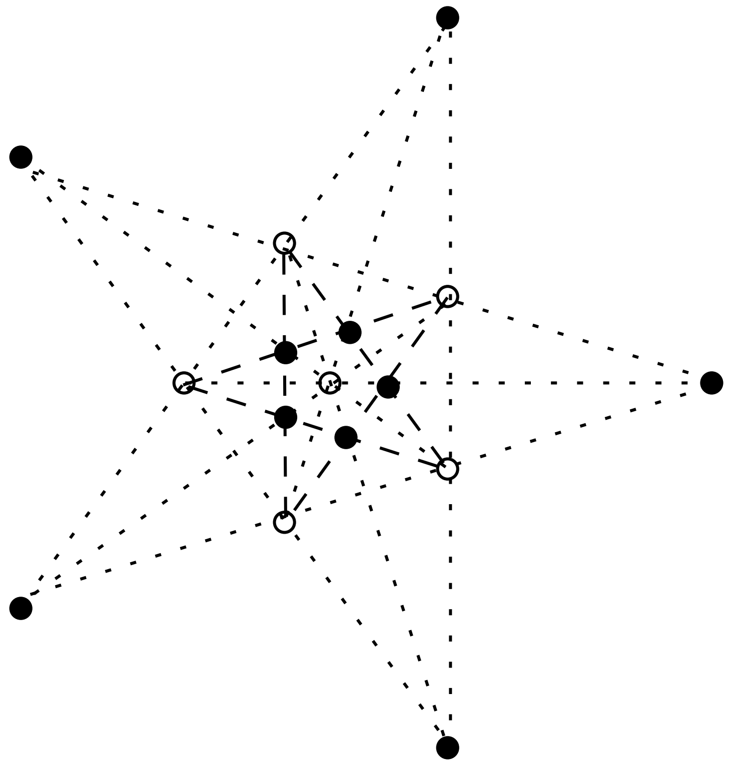

Figure 1: Pinchasi's set of six points with ten radial points

The notion of radiality can be extended, as follows. For any $\varepsilon \in (0,1)$, we say that $q \notin P$ is an *ε-radial point* for P, if the number of distinct lines connecting q to the points of P is at most $(1-\varepsilon)n$.

Note that, according to this definition, any radial point is (1/2)-radial. If at least $\varepsilon n + 1$ points of P lie on a common line, then every point of this line (which does not belong to P) is ε-radial. Therefore, in this case the number of ε-radial points for P is infinite. However, by a slight modification of the proof of Theorem 1, we can obtain the following positive result.

Theorem 2. *For any $0 < \delta < \varepsilon < 1$, there exists a constant $C = C(\delta, \varepsilon)$ with the following property.*

Let P be any set of n points in the plane. Then the number of ε-radial points that do not belong to any line passing through at least δn elements of P is at most Cn.

Corollary 3. *For any $0 < \delta < \varepsilon < 1$, there exists a constant $C = C(\delta, \varepsilon)$ with the following property. Let P be any set of n points in the plane, no δn of which are collinear. Then the number of ε-radial points for P is at most Cn.*

The following result of G. Elekes [3] (see also [4]) is another immediate consequence of Theorem 2.

Corollary 4. *For every $\varepsilon \leq 1$ there is a constant $C = C(\varepsilon)$ such that every n-element set P on a line has at most Cn subsets similar to a given $\lfloor \varepsilon n \rfloor$-element set Q.*

Proof: Assume without loss of generality that P and Q lie on two parallel lines in the plane. For any similar copy $Q' \subset P$ of Q, the lines connecting the corresponding points of Q and Q' are concurrent. Their common point is some $\varepsilon/2$-radial point for $P \cup Q$, which does not belong to any line containing more than two elements of this set. Hence, Theorem 2 can be applied to bound the number of such points (and sets Q'). $\square$

Note that Theorem 2 yields stronger results. For example, it shows that P has at most $C'n$ subsets, each similar to some subset of at least $\lfloor \varepsilon n/2 \rfloor$ elements of Q, for another constant $C' = C'(\varepsilon)$.

The proof of Theorem 1 is presented in Section 2. In Section 3, we establish a generalization of Theorem 2. The main result of that section, Theorem 3.1, is a strengthening of the following theorem of J. Beck [1] and E. Szemerédi and W.T. Trotter [9] (also known as the "weak Dirac-Motzkin conjecture").

Corollary 5. *There is a constant $c > 0$ with the property that any non-collinear set P of points in the plane has an element q such that the number of distinct lines connecting q to all other points of P is at least $c|P|$.*

According to the "strong Dirac-Motzkin conjecture," this statement should be true with $c = 1/2$. Corollary 5 has several interesting applications in combinatorial geometry (see e.g. [7], for the most recent one).

2 Proof of Theorem 1

Clearly, we only need to prove the upper bound. The idea of the proof is the following. Let q be a radial point for P, and let j denote the average number of points of P that lie on the lines connecting q to the points of P. If each of these lines contained exactly j points then the number of lines would be about n/j. In general, some relaxation of this relationship is needed: We show that any radial point q has an 'index' j such that the number of lines that connect q to at least j points of P is at least $n/(6j\log^2 j)$. We then show (in Lemma 2.4) that the number of radial points that have a small index (up to $c\sqrt{n}$ for some constant c) is linear, and finally derive a linear bound on the number of radial points with a large index.

The following well known results of Szemerédi and Trotter [9, 10] are crucial to the proof.

Lemma 2.1 *(i) The number of incidences between l distinct lines and n distinct points in the plane cannot exceed $3n^{2/3}l^{2/3} + n + l$.*

(ii) For any $j \leq n$, the number of lines containing at least j elements of a given set of n points in the plane cannot exceed $40 \cdot \max\{n^2/j^3,\ n/j\}$. $\square$

Part (i) is asymptotically tight in the worst case, apart from the values of the constants, and it implies part (ii). The best known constants for (i) are given in [5]. A simple proof of (i) was found by Székely [8]; see also [2].

In the sequel, let P be a fixed set of n points in the plane, not all on a line, and let R be the set of radial points for P.

Let $\mathcal{L}$ be the set of all lines that pass through at least two points of P. Denote by $\mathcal{L}_j$ (resp. $\mathcal{L}_{\le j}$, $\mathcal{L}_{\ge j}$) the set of those elements of $\mathcal{L}$ which contain precisely j (resp. at most j, at least j) elements of P. Throughout this paper, we write $\log j$ for $\ln j$, the natural logarithm of j.

Lemma 2.2 *For every radial point $q \in R$, there is an integer $2 \le j \le n-2$ such that the number of lines in $\mathcal{L}_{\ge j}$ passing through q is at least $\lceil \frac{n}{3j\log^2 j} \rceil$.*

Proof: Suppose to the contrary that for some $q \in R$ no such j exists. Let l_j denote the number of lines in $\mathcal{L}_j$ passing through q. We have

$$\begin{aligned} l_2 + l_3 + \cdots + l_{n-2} &< \tfrac{n}{3\cdot 2\log^2 2}, \\ l_3 + \cdots + l_{n-2} &< \tfrac{n}{3\cdot 3\log^2 3}, \\ &\cdots \\ l_{n-2} &< \tfrac{n}{3(n-2)\log^2(n-2)}. \end{aligned}$$

Summing up all these inequalities, the first one with coefficient two, and noting that $2l_2 + 3l_3 + \cdots + (n-2)l_{n-2} = n$, we obtain

$$n < n \cdot \left(\frac{1}{3\log^2 2} + \sum_{j=3}^{n-2} \frac{1}{3j\log^2 j} \right) < 0.8n,$$

the desired contradiction. □

Definition 2.3 *The* index *of a point $q \in R$ is the smallest integer $j = j(q)$ such that $2 \le j \le n-2$ and the number of lines in $\mathcal{L}_{\ge j}$ passing through q is at least $\lceil \frac{n}{6j\log^2 j} \rceil$.*

Clearly, Lemma 2.2 implies that every radial point has an index. (The constant 6 has been chosen for technical reasons that will become clear later.)

Lemma 2.4 *For every $c > 0$, there exists $c' > 0$ such that the number of radial points whose index is at most $c\sqrt{n}$ does not exceed $c'n$.*

Proof: Note first that Lemma 2.1(ii) implies that for $j \le \sqrt{n}$, the size of $\mathcal{L}_{\ge j}$ is at most $40n^2/j^3$, whereas for $j > \sqrt{n}$, the size of $\mathcal{L}_{\ge j}$ is at most $40n/j$. We will assume that $c > 1$; in the other case the proof gets only simpler.

For $j = 2, \ldots, \lfloor c\sqrt{n} \rfloor$, let R_j denote the set of radial points of index j, and put $r_j = |R_j|$. Let I_j denote the number of incidences between the points in R_j and the lines in $\mathcal{L}_{\ge j}$. Since each $q \in R_j$ is incident to at least $\frac{n}{6j\log^2 j}$ lines of $\mathcal{L}_{\ge j}$, we have

$$I_j \ge \frac{r_j n}{6j\log^2 j}.$$

Applying Lemma 2.1(i) to R_j and $\mathcal{L}_{\geq j}$, we obtain that

$$I_j \leq 3r_j^{2/3}|\mathcal{L}_{\geq j}|^{2/3} + r_j + |\mathcal{L}_{\geq j}|.$$

Comparing the last two inequalities, we have

$$\frac{r_j n}{6j\log^2 j} \leq 3r_j^{2/3} \cdot 40^{2/3}\left(\max\left\{\frac{n^2}{j^3}, \frac{n}{j}\right\}\right)^{2/3} + r_j + 40 \cdot \max\left\{\frac{n^2}{j^3}, \frac{n}{j}\right\}.$$

Therefore,

$$\frac{r_j n}{6j\log^2 j} \leq \max\left\{3^2 r_j^{2/3}\left(\frac{40n^2}{j^3}\right)^{2/3}, 3^2 r_j^{2/3}\left(\frac{40n}{j}\right)^{2/3}, 3r_j, \frac{120n^2}{j^3}, \frac{120n}{j}\right\},$$

which yields that, for $j \leq \sqrt{n}$,

$$r_j \leq \max\left\{6^3 \cdot 3^6 \cdot 40^2 \frac{n\log^6 j}{j^3}, 720\frac{n\log^2 j}{j^2}\right\} \leq 10^9\frac{n\log^2 j}{j^2},$$

and, for $j > \sqrt{n}$,

$$r_j \leq \max\left\{6^3 \cdot 3^6 \cdot 40^2 \frac{j\log^6 j}{n}, 720\log^2 j\right\} \leq 720\log^2 j,$$

provided that n is at least some sufficiently large constant n_0. Summing up these inequalities, we obtain that the number of radial points with index at most $c\sqrt{n}$ satisfies (for $n > n_0$)

$$\sum_{j=2}^{\lfloor c\sqrt{n}\rfloor} r_j \leq \sum_{j=2}^{\lfloor\sqrt{n}\rfloor} 10^9\frac{n\log^2 j}{j^2} + \sum_{j=\lfloor\sqrt{n}\rfloor+1}^{\lfloor c\sqrt{n}\rfloor} 720\log^2 j \leq c'n,$$

for an appropriate constant c'. (For $n \leq n_0$, this will trivially hold, if we choose c' sufficiently large.) $\Box$

Note that the dependence of c' on c is rather weak. In fact, if n is at least some constant $n_0(c)$ that depends on c, we can choose c' to be an absolute constant independent of c.

Let R^* denote the set of all radial points with index greater than $c\sqrt{n}$, where $c > 1$ is a constant to be specified later, and let $\mathcal{L}^* = \mathcal{L}_{\geq c\sqrt{n}}$. Then $R^* = R_1^* \cup R_2^*$, where R_1^* (resp. R_2^*) denotes the set of those elements of R^* that lie on exactly one line (resp. at least two lines) belonging to $\mathcal{L}^*$.

The number of incidences between the original point set P and $\mathcal{L}^*$ is at least $c\sqrt{n}|\mathcal{L}^*|$. On the other hand, by Lemma 2.1(i), the same quantity can be bounded from above by $3n^{2/3}|\mathcal{L}^*|^{2/3} + n + |\mathcal{L}^*|$. Thus, $|\mathcal{L}^*| < \sqrt{n}$, provided that c is sufficiently large ($c \geq 5$ will do). This immediately implies that

$$|R_2^*| \leq \binom{|\mathcal{L}^*|}{2} < n/2.$$

It remains to show that the size of R_1^* is $O(n)$.

Lemma 2.5 *For every point $q \in R_1^*$, the (unique) line in $\mathcal{L}^*$ passing through q contains more than $\lceil n/2 \rceil$ elements of P.*

Proof: Assume, in order to obtain a contradiction, that some $q \in R_1^*$ violates this condition. Then each line through q, different from the unique element ℓ^* of $\mathcal{L}^*$ that passes through q, contains at most $c\sqrt{n}$ points of P. (At this point we use the fact that the constant in the denominator of the quantity that appears in Definition 2.3 is 6.) Apply Lemma 2.2 to q and to the set $P' = P \setminus \ell^*$, to conclude that there exists an integer $2 \le j \le |P'| - 2$ such that q is incident to at least $|P'|/(3j \log^2 j)$ lines, each containing at least j points of P', and thus at least j points of P. Clearly, we have $j \le c\sqrt{n}$. Since $|P'| > n/2$, it follows, according to Definition 2.3, that the index of q (with respect to P) is at most $j \le c\sqrt{n}$. That is, $q \notin R^*$, a contradiction. □

Now we are in a position to complete the proof of Theorem 1. Since there are no two distinct lines passing through more than $\lceil n/2 \rceil$ points of P, it follows from Lemma 2.5 that all points of R_1^* are collinear. Let ℓ be the line containing R_1^*.

Let P' denote the set of points of P that do not lie on ℓ. For each radial point $q \in R_1^*$, let $f(q)$ denote the number of pairs $\{p', p''\} \subset P'$, for which q, p', and p'' are collinear. Obviously,

$$\sum_{q \in R_1^*} f(q) \le \binom{|P'|}{2}.$$

On the other hand, $f(q) \ge |P'|/2$ holds for every $q \in R_1^*$. This immediately implies that

$$|R_1^*| \le |P'| - 1 < n/2,$$

completing the proof of Theorem 1.

3 ε-Radial and Quasiradial Points

If a point $q \notin P$ is ε-radial, then the total number of points on those lines which pass through q and contain more than one element of P is at least εn. This suggests the following definition.

Let P be a set of n points in the plane, and let q be another point which may or may not belong to P. For any positive real $\varepsilon \le 1$ and for any integer $k \ge 2$, we say that q is an *(ε, k)-quasiradial point* for P, if the number of points $p \in P \setminus \{q\}$ sitting on lines that pass through q and at least k elements of $P \setminus \{q\}$ is at least εn.

Theorem 3.1 *For any $0 < \delta < \varepsilon \le 1$ and for any integer $k \ge 2$, there exists a constant $C = C(\delta, \varepsilon, k)$ with the following property.*

Let P be any set of n points in the plane. Remove from the plane all lines that pass through at least δn elements of P. Then the total number of (ε, k)-quasiradial points lying in the remaining regions is at most Cn, provided that n is large enough. Moreover, as n tends to infinity,

$$C(\delta, \varepsilon, k) = O\left(\frac{1}{(\varepsilon - \delta)^3 k \log k}\right).$$

The proof of this theorem is similar to that of Theorem 1.

Let P be a fixed set of n points, and let Q denote the set of all (ε, k)-quasiradial points for P. Then $Q = Q_0 \cup Q_1$, where $Q_0 = Q \setminus P$ and $Q_1 = Q \cap P$. Instead of Lemma 2.2, we now have

Lemma 3.2 *For every point $q \in Q_0$, there is an integer $j \geq k$, and, for every point $q \in Q_1$, there is an integer $j \geq k+1$ such that the number of lines in $\mathcal{L}_{\geq j}$ passing through q is at least $\lceil \frac{\varepsilon \log k}{3j \log^2 j} n \rceil$.*

Proof: Suppose to the contrary that for some $q \in Q$ no such j exists. Let l_j denote the number of lines in $\mathcal{L}_j$ passing through q, for $1 \leq j \leq n$. We have

$$\begin{array}{rcl} l_{k+1} + l_{k+2} + \cdots + l_n & < & \frac{\varepsilon \log k}{3(k+1) \log^2 (k+1)} \cdot n, \\ l_{k+2} + \cdots + l_n & < & \frac{\varepsilon \log k}{3(k+2) \log^2 (k+2)} \cdot n, \\ \cdots & & \\ l_n & < & \frac{\varepsilon \log k}{3 \cdot n \log^2 n} \cdot n, \end{array}$$

and, if $q \in Q_0$, then also

$$l_k + l_{k+1} + \cdots + l_n < \frac{\varepsilon \log k}{3k \log^2 k} \cdot n.$$

If $q \in Q_0$, then summing up these inequalities (the last one with coefficient k), and comparing it with the relation $kl_k + (k+1)l_{k+1} + \cdots \geq \varepsilon n$, we obtain

$$\varepsilon n < \varepsilon \log k \cdot \left(\frac{1}{3 \log^2 k} + \sum_{j=k+1}^{n} \frac{1}{3j \log^2 j} \right) \cdot n < 0.8 \varepsilon n,$$

the desired contradiction. If $q \in Q_1$, then $kl_{k+1} + (k+1)l_{k+2} + \cdots \geq \varepsilon n$. Hence, arguing as above, now we have

$$\varepsilon n < \varepsilon \log k \cdot \left(\frac{k}{3(k+1) \log^2 (k+1)} + \sum_{j=k+2}^{n} \frac{1}{3j \log^2 j} \right) \cdot n < 0.8 \varepsilon n,$$

again a contradiction. $\Box$

Definition 3.3 *The* index *of a point $q \in Q_0$ (resp. $q \in Q_1$) is the smallest integer $j = j(q) \geq k$ (resp. $j = j(q) \geq k+1$) such that the number of lines in $\mathcal{L}_{\geq j}$ passing through q is at least $\lceil \frac{(\varepsilon - \delta) \log k}{3j \log^2 j} \cdot n \rceil$.*

Lemma 3.2 implies that every point $q \in Q$ has a (unique) index.

Lemma 3.4 *Let $c > 1$ be fixed. There exists an absolute constant $d > 0$ (independent of c, δ, ε, and k) such that the number of points $q \in Q$ whose index is at most $c\sqrt{n}$ does not exceed $\frac{dn}{(\varepsilon - \delta)^3 k \log k}$, provided that $n \geq n_0(c, \delta, \varepsilon, k)$ is sufficiently large.*

Proof: For $j \geq k$, let $R_j \subseteq Q$ now denote the set of (ε, k)-quasiradial points of index j, and put $r_j = |R_j|$. Let I_j denote the number of incidences between the points in R_j and the lines in $\mathcal{L}_{\geq j}$. Since each $q \in R_j$ is incident to at least $\frac{(\varepsilon-\delta)\log k}{3j\log^2 j} \cdot n$ lines of $\mathcal{L}_{\geq j}$, we have

$$I_j \geq \frac{r_j(\varepsilon-\delta)\log k}{3j\log^2 j} \cdot n,$$

and, by Lemma 2.1(i),

$$I_j \leq 3r_j^{2/3}|\mathcal{L}_{\geq j}|^{2/3} + r_j + |\mathcal{L}_{\geq j}|.$$

Comparing the last two inequalities, as in the proof of Lemma 2.4, an easy computation shows that

$$\sum_{j=k}^{\lfloor c\sqrt{n}\rfloor} r_j \leq \sum_{j=k}^{\lfloor\sqrt{n}\rfloor} \frac{10^9 \log^2 j}{j^2(\varepsilon-\delta)^3\log^3 k} \cdot n + \sum_{j=\lfloor\sqrt{n}\rfloor+1}^{\lfloor c\sqrt{n}\rfloor} \frac{720\log^2 j}{(\varepsilon-\delta)\log k} \leq \frac{dn}{(\varepsilon-\delta)^3 k\log k},$$

for an appropriate absolute constant d, provided that n is sufficiently large. $\square$

It is interesting to note that we have not excluded in the proof of the lemma points that lie on 'heavy' lines, as prescribed in the statement of Theorem 3.1. In particular, the number of quasiradial points with a 'small' index is finite.

Let Q^* denote the set of all elements of Q which do not belong to any line containing at least δn points of P and whose index is greater than $c\sqrt{n}$, where we set

$$c = \max\{20, (\varepsilon-\delta)^{3/2}\sqrt{(5k/d)\log k}\},$$

with d being the same constant as in the previous lemma. Let $\mathcal{L}^* = \mathcal{L}_{\geq c\sqrt{n}}$. Then $Q^* = Q_1^* \cup Q_2^*$, where Q_1^* (resp. Q_2^*) denotes the set of those elements of Q^* which lie on exactly one line (resp. at least two lines) belonging to $\mathcal{L}^*$.

The number of incidences between the original point set P and $\mathcal{L}^*$ is at least $c\sqrt{n}|\mathcal{L}^*|$. On the other hand, by Lemma 2.1(i), the same quantity can be bounded from above by $3n^{2/3}|\mathcal{L}^*|^{2/3} + n + |\mathcal{L}^*|$. Thus, $|\mathcal{L}^*| < \sqrt{n}/c$. This immediately implies that

$$|Q_2^*| \leq \binom{|\mathcal{L}^*|}{2} < \frac{dn}{(\varepsilon-\delta)^3 k\log k},$$

if n is sufficiently large.

To complete the proof of Theorem 3.1, it is sufficient to establish the following.

Lemma 3.5 *Q_1^* is empty.*

Proof: Suppose, to the contrary, that Q_1^* has a point q. Let ℓ^* denote the unique line in $\mathcal{L}^*$ that passes through q, and let P' denote the set of points of P that lie outside ℓ^*. By assumption, ℓ^* contains at most δn points of P, so the size of P' is at least $(1-\delta)n$. Clearly, q is an $\left(\frac{(\varepsilon-\delta)n}{|P'|}, k\right)$-quasiradial point for P'. Applying Lemma 3.2 to P' and to q, we obtain that there is a $j \geq k$ such that the number of lines in $\mathcal{L}_{\geq j}$ passing through q (and excluding ℓ^*) is at least

$$\left\lceil \frac{(\varepsilon - \delta)\log k}{3j\log^2 j} \cdot n \right\rceil .$$

Hence, by Definition 3.3, the index of q is at most j. Since $q \in Q_1^*$, every line through q, except for ℓ^*, contains at most $c\sqrt{n}$ points of P. Thus, j and therefore the index of q cannot exceed $c\sqrt{n}$, contradicting our assumption that the index of all points in Q^* is larger than $c\sqrt{n}$. □

4 Concluding Remarks

4.1 Superradial points. Let P be a set of n points in the plane, not all on a line. We call a point $q \notin P$ *k-superradial* for P, if every line that passes through q and at least one point of P contains at least k elements of P. In this terminology, a radial point for P can also be called 2-superradial. Using the definition in the last section, a point is k-superradial if and only if it is $(1, k)$-quasiradial.

It follows from Theorem 3.1 that the number of k-superradial points for P is at most $c_k n$, with a coefficient c_k that goes to zero as k increases. The following construction shows that the the number of k-superradial points can exceed $c'_k\sqrt{n}$ for some $c'_k > 0$.

For any positive integer m, let $v_1^m, v_2^m, \ldots, v_M^m$ be the sequence of all vectors (p, q) with relatively prime integer coordinates satisfying $|p|, |q| \leq m$, listed in increasing order of their slopes (i.e., according to the clockwise angle between the positive x-axis and (p, q)). It is well known that $M > cm^2$ for a suitable constant $c > 0$. The points $w_j^m = \sum_{i=1}^{j} v_i^m$ $(j = 1, 2, \ldots, M)$ form the vertex set of a centrally symmetric convex polygon Q of perimeter at most $M\sqrt{2}m < 4\sqrt{2}m^3$. Let P be the set of all points $(x, y) \in Q$, whose distance from the boundary of Q is at most $2m$ and $(k-1)x$ and $(k-1)y$ are integers. Clearly, $|P| < 2(k-1)^2 M(2m) < c'(k-1)^2 m^4$, for some $c' > 0$. On the other hand, for every i $(1 \leq i \leq M)$, the intersection of the supporting line of v_i^m with the line at infinity is easily seen to be a k-superradial point for P. Therefore, the number of k-superradial points is at least

$$M > cm^2 > \frac{c''}{k-1}\sqrt{|P|},$$

for an appropriate constant $c'' > 0$, as asserted.

4.2 Lower bounds for quasiradial points. It can be shown by a similar construction that, for every fixed k and for every sufficiently large n, there exists a set P of n points in the plane with at most $o(n)$ points lying on any line, such that the number of $(1/2, k)$-quasiradial points for P is at least $\Omega(n/k^2)$. To see this, take all integer lattice points in a disk of radius $\sqrt{n/\pi}$, and notice that the intersection of the line at infinity with every line $y = (p/q)x$ with $|p|, |q| < \sqrt{n}/(10k)$ is a $(1/2, k)$-quasiradial point.

References

[1] J. Beck, On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry, *Combinatorica* **3** (1983), 281–297.

[2] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput. Geom.* **5** (1990), 99–160.

[3] G. Elekes, On the structure of large homothetic subsets, in: *Contemporary trends in discrete mathematics (Štiřmn Castle, 1997), DIMACS Ser. Discrete Math. Theoret. Comput. Sci.* **49**, Amer. Math. Soc., Providence, RI, 1999, 101–111.

[4] J. Pach and P.K. Agarwal, *Combinatorial Geometry,* John Wiley and Sons, New York, 1995.

[5] J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica* **17** (1997), 427–439.

[6] R. Pinchasi, On the size of a directed set of a set of points in the plane, preprint, Hebrew University, Jerusalem, 1999.

[7] J. Solymosi and Cs. Tóth, Distinct distances in the plane, *Discrete and Computational Geometry,* to appear.

[8] L.A. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combinatorics, Probability and Computing* **6** (1998), 353–358.

[9] E. Szemerédi and W.T. Trotter, Extremal problems in discrete geometry, *Combinatorica* **3** (1983), 381–392.

[10] E. Szemerédi and W.T. Trotter, A combinatorial distinction between the Euclidean and projective planes, *European J. Combinatorics* **4** (1983), 385–394.