A Tverberg-type result on multicolored simplices

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Abstract

Let $P_1, P_2, \dots, P_{d+1}$ be pairwise disjoint n-element point sets in general position in d-space. It is shown that there exist a point O and suitable subsets $Q_i \subseteq P_i$ $(i = 1, 2, \dots, d+1)$ such that $|Q_i| \geq c_d|P_i|$, and every d-dimensional simplex with exactly one vertex in each Q_i contains O in its interior. Here c_d is a positive constant depending only on d.

1 Introduction

Let $P_1, P_2, \dots, P_{d+1}$ be pairwise disjoint n-element point sets in general position in Euclidean d-space $\Re^d$. If two points belong to the same P_i, then we say that they are of the same *color*. A d-dimensional simplex is called *multicolored*, if it has exactly one vertex in each P_i $(i = 1, 2, \dots, d+1)$. Answering a question of Bárány, Füredi, and Lovász [BFL90], Vrećica and Živaljević [ZV92], proved the following Tverberg-type result. For every k, there exists an integer $n(k, d)$ such that if $n \geq n(k, d)$, then any pairwise disjoint n-element point sets $P_1, P_2, \dots, P_{d+1} \subset \Re^d$ in general position induce at least k multicolored vertex disjoint simplices with an interior point in common. (For some special cases, see [BL92], [JS91], [VZ94].) This theorem can be

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used to derive a nontrivial upper bound on the number of different ways one can cut a finite point set into two (roughly) equal halves by a hyperplane.

The aim of this note is to strengthen the above result by showing that there exist "large" subsets of the sets P_i such that *all* multicolored simplices induced by them have an interior point in common.

Theorem. *There exists $c_d > 0$ with the property that for any disjoint n-element point sets $P_1, P_2, \ldots, P_{d+1} \subset \Re^d$ in general position, one can find a point O and suitable subsets $Q_i \subseteq P_i$, $|Q_i| \geq c_d|P_i|$ $(i = 1, 2, \ldots, d+1)$ such that every d-dimensional simplex with exactly one vertex in each Q_i contains O in its interior.*

The proof is based on the $k = d+1$ special case of the Vrećica-Živaljević theorem (see Theorem 2.1). It uses three auxiliary results, each of them interesting on its own right. The first is Kalai's fractional Helly theorem [K84], which sharpens and generalizes some earlier results of Katchalski and Liu [KL79] (see Theorem 2.2). The second is a variation of Szemerédi's regularity lemma for hypergraphs [S78] (Theorem 2.3), and the third is a corollary of Radon's theorem [R21], discovered and applied by Goodman and Pollack [GPW96] (Theorem 2.4).

In the next section, we state the above mentioned results and also include a short proof of Theorem 2.3, because in its present form it cannot be found in the literature. Our argument is an adaptation of the approach of Komlós and Sós [KS96]. For some similar results, see [C91],[FR92],[KS95]. The proof of the Theorem is given in Section 3. It shows that the statement is true for a constant $c_d > 0$ whose value is triple-exponentially decreasing in d.

2 Auxiliary results

Theorem 2.1 [ZV92] *Let $A_1, A_2, \ldots, A_{d+1}$ be disjoint $4d$-element sets in general position in d-space. Then one can find $d+1$ vertex disjoint simplices with a common interior point such that each of them has exactly one vertex in every A_i, $1 \leq i \leq d+1$.*

A family of sets is called *intersecting* if they have an element in common.

Theorem 2.2 [K84] *For any $\alpha > 0$, there exists $\beta = \beta(\alpha, d) > 0$ satisfying the following condition. Any family of N convex sets in d-space, which contains at least $\alpha\binom{N}{d+1}$ intersecting $(d+1)$-tuples, has an intersecting subfamily with at least βN members.*

In fact, if N is sufficiently large, then Theorem 2.2 is true for any $\beta < 1-(1-\alpha)^{1/(d+1)}$. In particular, it holds for $\beta = \alpha/(d+1)$.

Let $\mathcal{H}$ be a $(d+1)$-partite hypergraph whose vertex set is the union of $d+1$ pairwise disjoint n-element sets, $P_1, P_2, \ldots, P_{d+1}$, and whose edges are $(d+1)$-tuples containing precisely one element from each P_i. For any subsets $S_i \subseteq P_i$ $(1 \le i \le d+1)$, let $e(S_1, \ldots, S_{d+1})$ denote the number of edges of $\mathcal{H}$ induced by $S_1 \cup \ldots \cup S_{d+1}$. In this notation, the total number of edges of $\mathcal{H}$ is equal to $e(P_1, \ldots, P_{d+1})$.

It is not hard to see that for any sets S_i and for any integers $t_i \le |S_i|$, $1 \le i \le d+1$,

$$\frac{e(S_1, \ldots, S_{d+1})}{|S_1| \ldots |S_{d+1}|} = \sum \frac{e(T_1, \ldots, T_{d+1})}{|T_1| \ldots |T_{d+1}|} \Big/ \binom{|S_1|}{t_1} \cdots \binom{|S_{d+1}|}{t_{d+1}}, \tag{1}$$

where the sum is taken over all t_i-element subsets $T_i \subseteq S_i$, $1 \le i \le d+1$.

Theorem 2.3 *Let $\mathcal{H}$ be a $(d+1)$-partite hypergraph on the vertex set $P_1 \cup \ldots \cup P_{d+1}$, $|P_i| = n$ $(1 \le i \le d+1)$, and assume that $\mathcal{H}$ has at least βn^{d+1} edges for some $\beta > 0$. Let $0 < \varepsilon < 1/2$.*

Then there exist subsets $S_i \subseteq P_i$ of equal size $|S_i| = s \ge \beta^{1/\varepsilon^{2d}} n$ $(1 \le i \le d+1)$ such that
(i) $e(S_1, \ldots, S_{d+1}) \ge \beta s^{d+1}$,
(ii) $e(Q_1, \ldots, Q_{d+1}) > 0$ for any $Q_i \subseteq S_i$ with $|Q_i| \ge \varepsilon s$ $(1 \le i \le d+1)$.

Proof: Let $S_i \subseteq P_i$ $(1 \le i \le d+1)$ be sets of equal size such that

$$\frac{e(S_1, \ldots, S_{d+1})}{|S_1|^{d+1-\varepsilon^{2d}}}$$

is *maximum*, and denote $|S_1| = \ldots = |S_{d+1}|$ by s.

For this choice of S_i, condition (i) in the theorem is obviously satisfied, because

$$\frac{e(S_1, \ldots, S_{d+1})}{|S_1|^{d+1-\varepsilon^{2d}}} \ge \frac{e(P_1, \ldots, P_{d+1})}{n^{d+1-\varepsilon^{2d}}} = \frac{\beta}{n^{-\varepsilon^{2d}}} \ge \frac{\beta}{s^{-\varepsilon^{2d}}}.$$

Taking into account the trivial relation

$$\frac{e(S_1,\ldots,S_{d+1})}{|S_1|^{d+1-\varepsilon^{2d}}} \le s^{\varepsilon^{2d}},$$

the above inequalities also yield that $s \ge \beta^{1/\varepsilon^{2d}} n$.

It remains to verify (ii). To simplify the notation, assume that εs is an integer, and let Q_i be any εs-element subset of S_i $(1 \le i \le d+1)$. Then

$$\begin{aligned} e(Q_1,\ldots,Q_{d+1}) &= e(S_1,\ldots,S_{d+1}) \\ &-e(S_1-Q_1,S_2,S_3,...,S_{d+1}) \\ &-e(Q_1,S_2-Q_2,S_3,...,S_{d+1}) \\ &-e(Q_1,Q_2,S_3-Q_3,...,S_{d+1}) \\ &\ldots \\ &-e(Q_1,Q_2,Q_3,...,S_{d+1}-Q_{d+1}). \end{aligned}$$

In view of (1), it follows from the maximal choice of S_i that

$$\begin{aligned} &e(S_1-Q_1,S_2,...,S_{d+1}) \\ &\quad = (1-\varepsilon)s^{d+1}\frac{e(S_1-Q_1,S_2,...,S_{d+1})}{|S_1-Q_1||S_2|...|S_{d+1}|} \\ &\quad = (1-\varepsilon)s^{d+1}\sum_{\substack{T_i\subseteq S_i,|T_i|=(1-\varepsilon)s\\ 2\le i\le d+1}}\frac{e(S_1-Q_1,T_2,...,T_{d+1})}{[(1-\varepsilon)s]^{d+1}}\Big/\binom{s}{\varepsilon s}^d \\ &\quad \le (1-\varepsilon)s^{d+1}\frac{e(S_1,S_2,...,S_{d+1})}{s^{d+1-\varepsilon^{2d}}}[(1-\varepsilon)s]^{-\varepsilon^{2d}} \\ &\quad = e(S_1,...,S_{d+1})(1-\varepsilon)^{1-\varepsilon^{2d}}. \end{aligned}$$

Similarly, for any $i,\ 2\le i\le d+1$, we have

$$\begin{aligned} &e(Q_1,...,Q_{i-1},S_i-Q_i,S_{i+1},...,S_{d+1}) \\ &\quad \le e(S_1,...,S_{d+1})\varepsilon^{i-1-\varepsilon^{2d}}(1-\varepsilon). \end{aligned}$$

Summing up these inequalities, we obtain

$$e(Q_1,\ldots,Q_{d+1}) \geq e(S_1,\ldots,S_{d+1})(1-(1-\varepsilon)^{1-\varepsilon^{2d}} - \sum_{i=2}^{d+1} \varepsilon^{i-1-\varepsilon^{2d}}(1-\varepsilon))$$
$$\geq e(S_1,\ldots,S_{d+1})(1-(1-\varepsilon)^{1-\varepsilon^{2d}} - \varepsilon^{1-\varepsilon^{2d}} + \varepsilon^{d+1-\varepsilon^{2d}}) > 0,$$

as required. □

A $(d+1)$-tuple of convex sets in d-space is called *separated* if any j of them can be strictly separated from the remaining $d+1-j$ by a hyperplane, $1 \leq j \leq d$. An arbitrary family of at least $d+1$ convex sets in d-space is *separated* if every $(d+1)$-tuple of it is separated.

Theorem 2.4 [GPW96] *A family of convex sets in d-space is separated if and only if no $d+1$ of its members can be intersected by a hyperplane.*

Let $n \geq d+1$. Two sequences of points in d-space, $(p_1,\ldots,p_n)$ and $(q_1,\ldots,q_n)$, are said to have the same *order type* if for any integers $1 \leq i_1 < \ldots < i_{d+1} \leq n$, the simplices $p_{i_1}\ldots p_{i_{d+1}}$ and $q_{i_1}\ldots q_{i_{d+1}}$ have the same orientation [GPW93]. It readily follows from the last result that if $C_1,\ldots,C_n$ form a separated family of convex sets, then the order type of $(p_1,\ldots,p_n)$ will be the same for every choice of elements $p_i \in C_i$, $1 \leq i \leq n$.

3 Proof of Theorem

Let $P_1,\ldots,P_{d+1}$ be pairwise disjoint n-element point sets in general position in d-space. If a simplex has precisely one vertex in each P_i, we call it *multicolored.* The number of multicolored simplices is $N = n^{d+1}$.

By Theorem 2.1, any collection of $4d$-element subsets $A_i \subseteq P_i$, $1 \leq i \leq d+1$, induce $d+1$ vertex disjoint multicolored simplices with a common interior point. Thus, the total number of intersecting $(d+1)$-tuples of multicolored simplices is at least

$$\frac{\binom{n}{4d}^{d+1}}{\binom{n-d-1}{3d-1}^{d+1}} > \frac{1}{(5d)^{d^2}}\binom{N}{d+1}.$$

Hence, we can apply Theorem 2.2 with $\alpha = 1/(5d)^{d^2}$. We obtain that there is a point O contained in the interior of at least $\beta N = \beta(1/(5d)^{d^2}, d)n^{d+1}$ multicolored simplices.

Let $\mathcal{H}$ denote the $(d+1)$-partite hypergraph on the vertex set $P_1 \cup \ldots \cup P_{d+1}$, whose edge set consists of all multicolored $(d+1)$-tuples that induce a simplex containing O in its interior.

Set $\varepsilon = 1/2^{d2^d}$, and apply Theorem 2.3 to the hypergraph $\mathcal{H}$ to find $S_i \subseteq P_i$, $1 \le i \le d+1$, meeting the requirements. By throwing out some points from each S_i, but retaining a positive proportion of them, we can achieve that the convex hulls of the sets S_i are separated. Indeed, assume e.g. that there is no hyperplane strictly separating $S_1 \cup \ldots \cup S_j$ from $S_{j+1} \cup \ldots \cup S_{d+1}$. By the *ham-sandwich theorem* [B33], one can find a hyperplane h which simultaneously bisects $S_1, \ldots, S_d$ into as equal parts as possible. Assume without loss of generality that at least half of the elements of S_{d+1} are "above" h. Then throw away all elements of $S_1 \cup \ldots \cup S_j$ that are above h and all elements of $S_{j+1} \cup \ldots \cup S_{d+1}$ that are below h. We can repeat this procedure as long as we find a non-separated $(d+1)$-tuple. In each step, we reduce the size of every set by a factor of at most 2.

Notice that in the same manner we can also achieve that e.g. the $(d+1)$-tuple $\{\{O\}, \text{conv}(S_1), \ldots, \text{conv}(S_d)\}$ becomes separated. In this case, h will always pass through the point O, therefore O will never be deleted.

After at most $(d+2)2^d$ steps we end up with $Q_i \subseteq S_i$, $|Q_i| > \varepsilon s$ $(1 \le i \le d+1)$ such that $\{\{O\}, \text{conv}(S_1), \ldots, \text{conv}(S_{d+1})\}$ is a separated family. It follows from the remark after Theorem 2.4 that there are only two possibilities: either every multicolored simplex induced by $Q_1 \cup \ldots \cup Q_{d+1}$ contains O in its interior, or none of them does. However, this latter option is ruled out by part (ii) of Theorem 2.3. This completes proof. $\Box$

Instead of applying Theorem 2.2, we could have started the proof by referring to the following result of Alon, Bárány, Füredi, and Kleitman [ABFK92], which is also based on Theorem 2.1. For any $\beta > 0$ there is a $\beta_d' > 0$ such that any family of βn^{d+1} simplices induced by n points in d-space has at least $\beta_d' n^{d+1}$ members with non-empty intersection.

Our proof easily yields the following.

Theorem 3.1 *For any $\beta > 0$ there is a $\beta_d' > 0$ with the property that given any family $\mathcal{S}$ of βn^{d+1} simplices induced by an n-element set $P \subset \Re^d$, one can find a point O and pairwise disjoint subsets $Q_i \subseteq P$ $(i = 1, 2, \ldots, d+1)$ such that every d-dimensional simplex with exactly one vertex in each Q_i contains O, and at least $\beta_d' n^{d+1}$ of them belong to $\mathcal{S}$.*

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