Embedding Planar Graphs at Fixed Vertex Locations

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Abstract. Let G be a planar graph of n vertices, $v_1, \ldots, v_n$, and let $\{p_1, \ldots, p_n\}$ be a set of n points in the plane. We present an algorithm for constructing in $O(n^2)$ time a planar embedding of G, where vertex v_i is represented by point p_i and each edge is represented by a polygonal curve with $O(n)$ bends (internal vertices.) This bound is asymptotically optimal in the worst case. In fact, if G is a planar graph containing at least m pairwise independent edges and the vertices of G are randomly assigned to points in convex position, then, almost surely, every planar embedding of G mapping vertices to their assigned points and edges to polygonal curves has at least $m/20$ edges represented by curves with at least $m/40^3$ bends.

1 Introduction

A *planar embedding* of a planar graph G is a drawing of G in the plane, where the edges are represented by simple continuous curves which intersect only at endpoints representing common vertices. By definition, every planar graph has a planar embedding. In 1948, Fáry showed that every planar graph has a *straight-line embedding*, i.e., a planar embedding where every edge is represented by a single line segment. Numerous algorithms exist for constructing straight-line embeddings. These algorithms construct and report a set of locations for the vertices of G. The line segments representing the edges of G are implicitly given by the locations of their endpoints.

In this paper, we consider the problem of constructing a planar embedding where vertices are mapped to prespecified fixed locations. If $\{v_1, \ldots, v_n\}$ is the vertex set of G and $\{p_1, \ldots, p_n\}$ is the set of vertex locations, then each v_i must be mapped to the point p_i. In most cases, the edges of G can no longer be represented by disjoint line segments. However, they can be represented by non-crossing polygonal curves. What is the complexity of these polygonal curves, i.e., how many *bends* (internal vertices) must they have? By a slight modification of the algorithm presented in Souvaine and Wenger [9], we obtain that if G is a

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single path, then it has a planar embedding with a total of $O(n^2)$ bends. Here we prove a stronger result, which answers a question asked by Richard Pollack.

Theorem 1. *Every planar graph on n vertices admits a planar embedding which maps each vertex to an arbitrarily prespecified distinct location and each edge to a polygonal curve with $O(n)$ bends.*

Moreover, such an embedding can be constructed in $O(n^2)$ time.

We apply an idea of Pach, Shahrokhi, and Szegedy [8] to show that the bound in Theorem 1 is tight. We say that two edges of a graph are *independent*, if they do not share an endpoint. A set of pairwise independent edges is often called a *matching*. A set of $n/2$ pairwise independent edges in a graph with n vertices is called a *perfect matching*.

Theorem 2. *Let G be a planar graph of n vertices, $v_1, \ldots, v_n$, which contains at least m pairwise independent edges and let $(p_1, \ldots, p_n)$ be a random permutation of the vertices of a convex n-gon.*

Then, as n tends to infinity, in every planar embedding of G which maps v_i to p_i and the edges to polygonal curves, there are almost surely at least $m/20$ edges represented by curves with at least $m/40^3$ bends.

Note that a path of length n or a perfect matching of n vertices have $\lfloor n/2 \rfloor$ pairwise independent edges.

The nature of the problem drastically changes if we only require that $\{v_1, \ldots, v_n\}$ be mapped to a set $\{p_1, \ldots, p_n\}$ of points in general position but we do not insist on the particular order. It is known that, if G is an outerplanar graph, there always exists a straight-line embedding which maps v_i to $p_{\pi(i)}$, $1 \leq i \leq n$, for a suitable permutation π (see [2], [7]).

There has also been extensive work on embedding a planar graph in a rectangular 2-dimensional grid so that graph vertices map to grid vertices and graph edges map to internally disjoint paths in the grid[5]. The maximum degree of any vertex in the graph must be four for such an embedding to exist. The objective is to minimize the number of bends in the paths representing graph edges. An embedding with minimum number of bends can be found in $O(n^2 \log n)$ time[10] and an embedding with at most two bends per edge can be found in linear time[6] (except on one special graph.) The location of two vertices of the graph can even be fixed[3] but the maximum number of bends per edge increases to four. However, in this paper the location of all the vertices are fixed, causing some paths to require at least $m/40^3$ bends.

2 Embedding Algorithm – Proof of Theorem 1

Let G be a planar graph with vertex set $V = \{v_1, \ldots, v_n\}$, and let $\{p_1, \ldots, p_n\}$ be a set of n points in the plane. In this section, we give an algorithm for constructing a planar embedding of G such that vertex v_i is represented by point p_i and each edge of G is represented by a polygonal curve with $O(n)$ bends (internal vertices.) Our algorithm runs in $O(n^2)$ time.

A *Hamiltonian cycle* is a cycle which visits each vertex of the graph exactly once. A planar embedding of a cycle divides the plane into a bounded and an unbounded component. If the edges are represented by polygonal curves, then the bounded component is a simple polygon.

The general outline of our algorithm is as follows. First, we establish an easy upper bound on the number of bends in a polygonal curve which follows the boundary of a tree at constant distance under the l_1 metric (Lemma 3). We next assume that G contains a Hamiltonian cycle and show how to construct a planar embedding of G mapping each vertex v_i to point p_i (Lemma 4). Finally, we show how to add vertices and edges to a planar graph forming a new planar graph which has a Hamiltonian cycle (Lemma 5). Combining the last two results gives a proof of Theorem 1.

To construct the planar embedding of a graph containing a Hamiltonian cycle, we construct a polygonal, planar embedding of a tree whose leaves are the points $p_1, \ldots, p_n$. Curves representing the edges of the cycle will follow the boundary of this tree or one of its subtrees at a fixed distance. We use the l_1 metric to measure this distance. The distance between points (x_1, y_1) and (x_2, y_2) in the l_1 metric is $|x_1 - x_2| + |y_1 - y_2|$. Let B_ϵ be the ball of radius ϵ in the l_1 metric. The *Minkowski sum* of a point set S and B_ϵ, written $S + B_\epsilon$, is the set of all points which are at distance less than or equal to ϵ from some point in S.

Lemma 3. *Let T be a straight-line embedding of a tree with $N > 1$ vertices. If $\epsilon > 0$ is smaller than the l_1-distance between any line segment (edge) e and any vertex of T not incident to e, then $T + B_{\epsilon/2}$ is a simple polygon whose boundary contains at most $4N - 2$ vertices.*

Proof. Each vertex of the polygon $T + B_{\epsilon/2}$ is either a vertex of $u + B_{\epsilon/2}$ for some vertex u of T or lies on the intersection of the boundary of $e + B_{\epsilon/2}$ and $e' + B_{\epsilon/2}$ for two adjacent edges e, e' of T. The adjacent edges e and e' share some common endpoint u. Thus each vertex of the polygon $T + B_{\epsilon/2}$ can be associated with a vertex u of T. The number of vertices associated with u is at most $2 + d_u$ where $d_u \geq 1$ is the degree of u. A tree has $N - 1$ edges so the sum of the degrees of all tree vertices is $2N - 2$. Thus summing $2 + d_u$ over all vertices of the tree gives a bound of $4N - 2$ on the number of vertices of $T + B_{\epsilon/2}$. □

An embedding of a graph is *outerplanar* if it is a simple closed curve with some non-crossing internal diagonals. The unbounded face of such an embedding corresponds to the (uniquely determined) Hamiltonian cycle of the graph. A graph is *outerplanar* if it has an outerplanar embedding. Mapping the vertices of an outerplanar graph to vertices of a convex n-gon in the order (clockwise or counter-clockwise) that they appear in the Hamiltonian cycle and its edges to line segments between n-gon vertices gives a straight-line planar embedding of the outerplanar graph. Every planar graph containing a Hamiltonian cycle can be divided into two outerplanar graphs which have only the edges of the Hamiltonian cycle in common.

We are ready to construct the planar embedding of a graph containing a Hamiltonian cycle.

Lemma 4. *Let G be a planar graph of n vertices, $v_1, \ldots, v_n$, containing a Hamiltonian cycle C and let $\{p_1, \ldots, p_n\}$ be a set of n distinct points in the plane. Graph G has a planar embedding such that*

(i) every vertex v_i of G is represented by the point p_i;

(ii) every edge of G is represented by a polygonal curve with at most $8n+9$ bends.

Moreover, such an embedding can be constructed in $O(n^2)$ time.

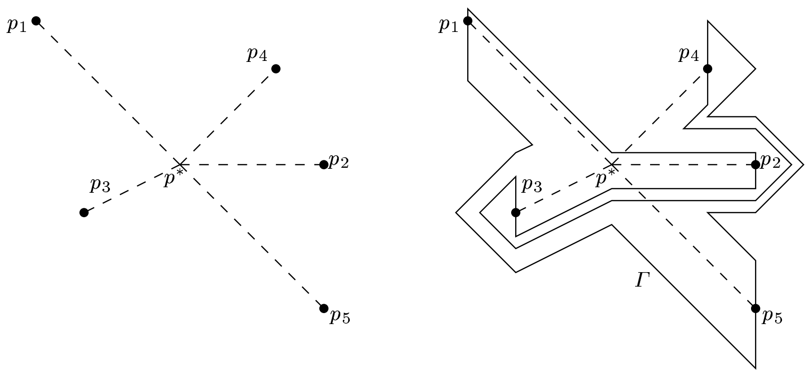

Fig. 1. Tree T and the simple polygonal curve Γ through $p_1, \ldots, p_5$.

Proof. By relabeling vertices and points, we may assume that the vertices of G appear in the Hamiltonian cycle C in the order $(v_1, \ldots, v_n)$. We first construct an embedding of the edges of C.

Let p^* be some point in the plane to the right of p_1 and in general position with respect to all the points p_i, i.e., the line through any two distinct points, p_i and p_j, does not contain p^*. In addition, p^* should not share an x-coordinate with any of the points p_i. Let T_1 be the line segment (p^*, p_1) and T_i be the union of T_{i-1} and (p^*, p_i). Set T equal to T_n. Note that the leaves of T_i are $p_1, \ldots, p_i$ and that $T_1 \subset T_2 \subset \cdots \subset T_n = T$. The edges of the cycle will be routed around the embedded trees T_i.

Let $\epsilon > 0$ be the smallest distance under the l_1 metric between any p_i and any line segment (p^*, p_j), where $j \neq i$. Let δ equal $\epsilon/(4n)$. Draw a simple closed polygonal curve Γ through the points $p_1, \ldots, p_n$ in sequential order as follows. Start by drawing a vertical line segment from p_1 to a point at distance δ above p_1. ¿From this point, draw a polygonal curve clockwise along the boundary of $T_2 + B_\delta$ where B_δ is the ball of radius δ in the l_1 metric. Stop when this polygonal curve through one of the two points at distance δ above or below p_2 and draw a vertical line segment of length 3δ through p_2. The endpoint of the curve is now a distance 2δ from p_2.

For $i = 2$ to $n-1$, draw a polygonal curve along the boundary of $T_{i+1} + B_{i\delta}$. Stop when this polygonal curve through one of the two points at distance $i\delta$ above or below p_{i+1} and draw a vertical line segment of length $(2i+1)\delta$ through p_{i+1}. There are two possible such curves, one clockwise and one counter-clockwise around T_{i+1}. Choose the one which does not go around p_1, i.e., which does not contain the point which is distance $i\delta$ directly to the left of p_1. Note that since $T_{j+1} + B_{j\delta}$ lies in the interior of $T_{i+1} + B_{i\delta}$ for any $j < i$, the curve will not intersect any of the previously drawn curves.

Complete Γ by drawing a polygonal curve clockwise around T at a distance of $n\delta$ to the point which is a distance $n\delta$ directly below p_1 and finally connecting this point to p_1 by a vertical line segment. (See Figure 1.) The simple closed polygonal curve Γ will be the image of the Hamiltonian cycle C in our embedding. Note that p^* lies in the bounded region defined by Γ and that the points $p_1, \ldots, p_n$ lie in clockwise order around Γ.

Except for the two line segments on either end, the curve from p_i to p_{i+1} lies on the boundary of $T_{i+1} + B_{i\delta}$. Tree T_{i+1} is a subtree of T which has at most $n+1$ vertices. By Lemma 3, the curve from p_i to p_{i+1} has at most $(4(n+1)-2)+2 = 4(n+1)$ bends. Similarly, the curve from p_n to p_1 has at most $4(n+1)$ bends.

We now construct pairwise disjoint auxiliary paths between the points p_i and p_1. For positive integers $i \leq n$ and $k \leq n-1$, let $\Lambda_{i,k}$ be the boundary of $T_i + B_{(i-1)\delta + k\delta/n}$. The polygonal curve $\Lambda_{i,k}$ intersects Γ at only two places, once on a line segment incident with p_i and once on a line segment incident with p_1. (See Figure 2.) Thus, Γ divides $\Lambda_{i,k}$ into two polygonal curves, one in the bounded region and one in the unbounded region defined by Γ. Truncate these curves at a distance δ from their endpoints and connect their new endpoints to p_i and p_1 by line segments, thus forming two paths between p_i and p_1. One of these paths lies in the interior of the bounded region and one in the interior of the unbounded region defined by Γ. Label these paths $\Lambda'_{i,k}$ and $\Lambda''_{i,k}$ in the bounded and unbounded regions, respectively. These paths are pairwise disjoint except perhaps at their endpoints. Applying Lemma 3, shows that each of these paths have at most $4(n+1)$ bends.

We are finally ready to embed the remaining edges of G. Divide G into two outerplanar graphs G' and G'' which have only the edges of Hamiltonian cycle C in common. Let E' and E'' be the edges of G' and G'', respectively, which are not in C.

We represent each edge (v_i, v_1) in E' by the path $\Lambda'_{i,1}$. Similarly, we represent each edge (v_i, v_1) in E'' by the path $\Lambda''_{i,1}$. We represent edges (v_i, v_j) where $i, j > 1$ as follows.

Draw a circle Φ of radius $\delta/2$ in the Euclidean norm around p_1 and intersect it with each of the $\Lambda'_{i,k}$ and $\Lambda''_{i,k}$. Since the paths from p_1 to p_2 and p_n start with vertical line segments above and below p_1, the curve Γ divides Φ into two

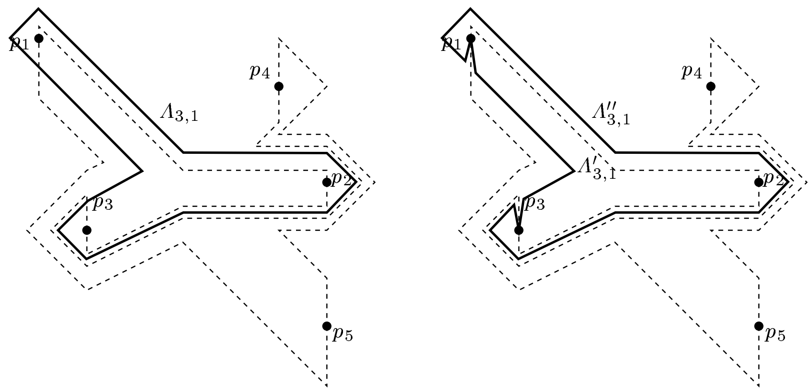

Fig. 2. The polygonal curve $\Lambda_{3,k}$ and paths $\Lambda'_{3,k}$ and $\Lambda''_{3,k}$.

semi-circles, Φ' and Φ'' in the bounded and unbounded regions defined by Γ. Let $q'_{i,k}$ and $q''_{i,k}$ be the intersection points of $\Lambda'_{i,k}$ and $\Lambda''_{i,k}$ with Φ, respectively.

For integers i, j, let $f(i,j)$ be the unique non-negative integer less than n which is congruent to $(i-j) \bmod n$. Represent edge $(v_i, v_j) \in E'$ by a path from p_i along $\Lambda'_{i,f(i,j)}$ to $q'_{i,f(i,j)}$, then a line segment from $q'_{i,f(i,j)}$ to $q'_{j,f(j,i)}$, and finally a path along $\Lambda'_{j,f(j,i)}$ from $q'_{j,f(j,i)}$ to p_j. This path contains at most $4(n+1)+4(n+1)+1 = 8n+9$ bends. (See Figure 3 for an example of a complete embedding of a graph.)

We claim that this embedding of G' is planar. By construction, the paths $\Lambda'_{i,k}$ intersect or touch Γ only at their endpoints. Thus the only possible violations of planarity occur in the line segments that we drew in Φ'.

Since G' is outerplanar, mapping its vertices to vertices of a convex n-gon and its edges to a line segments between vertices gives a planar embedding of G'. The points on Φ' form vertices of a convex polygon and lie in the clockwise order

$$p_1, q'_{2,1}, \ldots, q'_{2,n-1}, q'_{3,1}, \ldots, q'_{3,n-1}, \ldots, q'_{n,1}, \ldots, q'_{n,n-1}.$$

For each edge (v_i, v_j) of E', we drew the line segment $(q'_{i,f(i,j)}, q'_{j,f(j,i)})$ in Φ'. If instead we drew the line segment $(q'_{i,1}, q'_{j,1})$ in the convex n-gon $(p_1, q'_{1,1}, q'_{2,1}, \ldots, q'_{n,1})$ for each $(v_i, v_j) \in E'$, we would have a planar embedding of G'. If (v_i, v_j) and $(v_{i'}, v_{j'})$ are two edges in E' and $v_i, v_j, v_{i'}$ and $v_{j'}$ are distinct, then $q'_{i,f(i,j)}$, $q'_{j,f(j,i)}$, $q'_{i',f(i',j')}$ and $q'_{j',f(j',i')}$ lie in the same order around Φ' as $q'_{i,1}$, $q'_{j,1}$, $q'_{i',1}$, $q'_{j',1}$. Since line segments $(q'_{i,1}, q'_{j,1})$ and $(q'_{i',1}, q'_{j',1})$ are pairwise disjoint, so are line segments $(q'_{i,f(i,j)}, q'_{j,f(j,i)})$ and $(q'_{i',f(i',j')}, q'_{j',f(j',i')})$. If (v_i, v_j) and $(v_i, v_{j'})$ are two edges in E' sharing the vertex v_i, then the relative positions on Φ' of $q'_{i,f(i,j)}$ and $q'_{i,f(i,j')}$ ensure that $(q'_{i,f(i,j)}, q'_{j,f(j,i)})$ and $(q'_{i,f(i,j')}, q'_{j',f(j',i)})$ are pairwise disjoint.

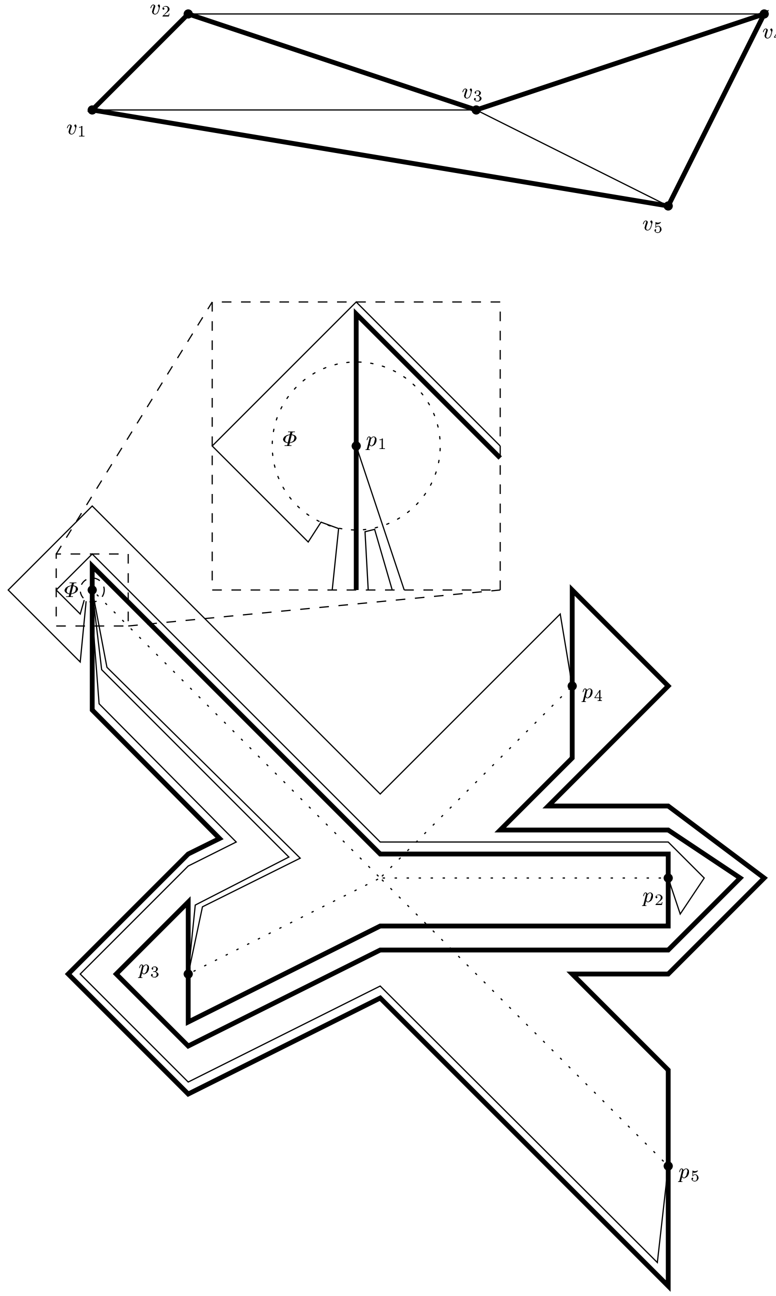

Fig. 3. Planar embedding of graph G where vertex v_i maps to point p_i.

An edge $(v_i, v_j) \in E''$ is represented by a path from p_i along $\Lambda''_{i,f(i,j)}$ to $q''_{i,f(i,j)}$, followed by a line segment from $q''_{i,f(i,j)}$ to $q''_{j,f(j,i)}$, and finally a path from $q''_{j,f(j,i)}$ to p_j. This path also contains at most $4(n+1)+4(n+1)+1 = 8n+9$ line bends. A similar argument to the one for E' shows that this embedding of E'' is planar. The only difference is that the points lie in counterclockwise order around Φ''.

Finding a point p^* in general position and calculating the value of ϵ can be done in $O(n \log n)$ time. Constructing the polygonal curves from p_{i-1} to p_i and from p_n to p_1 takes $O(n)$ time per curve. Constructing the polygonal curves representing edges in E' and E'' also take $O(n)$ time per edge for a total of $O(n^2)$ time. □

We now show how to turn a planar graph G into a planar graph H containing a Hamiltonian cycle.

Lemma 5. *Let G be a planar graph of n vertices. By subdividing edges of G by at most two new vertices and adding some edges between vertices, we can construct from G a planar graph H which has at most $5n-10$ edges and contains a Hamiltonian cycle. This construction can be accomplished in linear time.*

Proof. If G has $k > 1$ connected components, add $k-1$ new edges to form a connected graph G'. Let S be a spanning tree of G'. Clearly, all edges of $G' - G$ belong to S. (See Figure 4.)

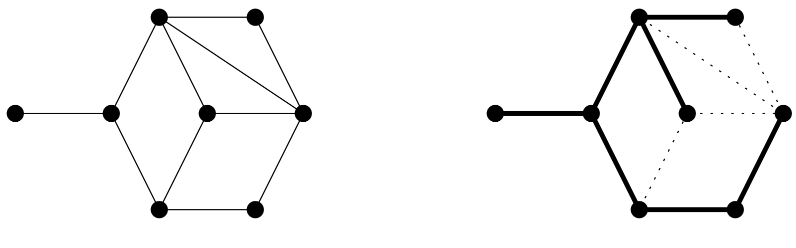

Fig. 4. Graph G and a spanning tree S.

Construct any planar embedding (not necessarily a straight-edge one) of G'. Starting at any vertex, walk clockwise around S, visiting its vertices in order. Note that the internal vertices of S will be visited more than once. Label the vertices with $w_1, w_2, \ldots, w_n$, by the order, in which they are first visited. If w_i and w_{i+1} are connected by an edge, then let this edge belong to the Hamiltonian cycle ($1 \le i \le n$, and we use the convention $w_{n+1} := w_1$). If not, connect w_i to w_{i+1} by a simple curve clockwise around the boundary of S, passing very close to it. Wherever this curve intersects an edge of G', introduce a new vertex. Thus, this curve becomes a path whose pieces are added as edges to the graph

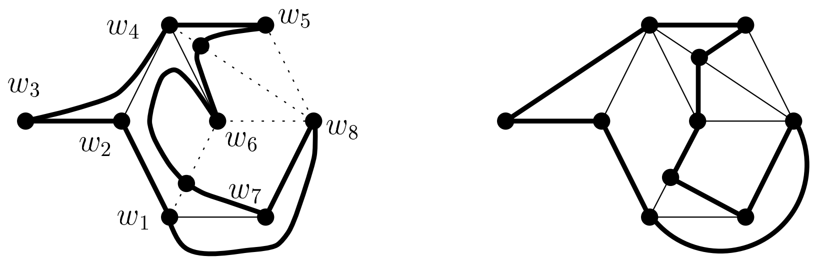

Fig. 5. Constructing the Hamiltonian cycle and graph H.

and to its Hamiltonian cycle. Merge any multiple edges, and call the resulting graph H. (See Figure 5.)

Each edge of G' was split at most twice. Since G' has at most $3n-6$ edges and the $n-1$ edges of S were never split, H has at most $n+2(2n-5) = 5n-10$ vertices. Connecting G to form G', constructing a planar embedding and a spanning tree S of G' take $O(n)$ time. Walking around S to construct the Hamiltonian cycle can also be done in linear time. □

Proof of Theorem 1: Starting with a planar graph G of n vertices, form a planar graph H on at most $5n-10$ vertices containing a Hamiltonian cycle as outlined in Lemma 5. Only the n original vertices from G have prespecified locations, but we can arbitrarily assign locations to the new vertices in H. Applying Lemma 4 gives a planar embedding of H, whose edges are polygonal curves with at most $8(5n-10)+9 < 40n$ bends. We construct an embedding of G from the embedding of H by merging polygonal curves which correspond to portions of edges of G and by deleting polygonal curves which are not part of any edge of G. Each edge of G was split into at most 3 edges in H, so each edge in G can be represented by a polygonal curve with at most $3 * 40n = 120n$ bends.

Constructing H takes $O(n)$ time. Embedding H in the plane takes $O(n^2)$ time. Constructing the embedding of G from the embedding of H takes time proportional to the size of the embedding of H which is $O(n^2)$. Thus, our algorithm runs in $O(n^2)$ time. □

A more generalized version of Lemma 4 is also true. Given a Hamiltonian, planar graph H' of n vertices where only $k \geq 1$ vertices of H' have preassigned point locations, it is possible to represent every edge of H' by a polygonal curve with at most $8k+9$ bends. Applying this bound to H in the proof of Theorem 1 gives a planar embedding of H with at most $8n+9 < 8(n+2)$ bends per curve. This results in each edge of the original graph G being represented by curves with at most $3 * 8(n+2) = 24(n+2)$ bends.

3 Lower Bound – Proof of Theorem 2

Let H be a graph with vertex set $V(H)$ and edge set $E(H)$. The *bisection width* $b(H)$ of H is the minimum number of edges running between V_1 and V_2, over all partitions of the vertex set $V(H)$ into two parts $V_1 \cup V_2$ such that $|V_1|, |V_2| \geq |V(H)|/3$. Roughly speaking, the bisection width is the minimum number of edges whose deletion splits the graph into two approximately equal parts.

The *crossing number* $c(H)$ of H is the minimum number of crossing pairs of arcs in any planar representation of H, where the vertices are mapped to distinct points and the edges are represented by simple continuous arcs connecting the corresponding points and not passing through the image of any other vertex.

Leighton [4] discovered that the above parameters are closely related. A somewhat more general form of his result was established in [8].

Lemma 6. *Let H be a graph of n vertices, whose degrees are $d_1, \dots, d_n$. Then*

$$b^2(G) \leq (1.58)^2 \left(16c(G) + \sum_{i=1}^{n} d_i^2\right).$$

The following lemma bounds the bisection width of a random graph which is constructed from the union of a cycle and a random perfect matching.

Lemma 7. *Let H_n be a randomly defined graph with vertex set $V(H_n) = \{u_1, \dots, u_n\}$, whose edge set is the union of $\{u_1u_2, u_2u_3, \dots, u_nu_1\}$ and a random perfect matching of the vertices (n is even).*

Then, as n tends to infinity, the bisection width of H_n is almost surely at least $n/20$.

Proof. The number of different graphs that can be obtained as H_n is equal to the number of perfect matchings on n vertices, $\frac{n!}{(n/2)!2^{n/2}}$. All of these graphs are equally likely.

Next we estimate that at most how many of these graphs have bisection width at most k, for some fixed $k \leq n/3$. Consider the cycle $C = u_1 ... u_n \subseteq H_n$. The number of partitions $V(H_n) = V_1 \cup V_2$, for which at most k edges of C run between V_1 and V_2, is at most

$$\sum_{i=0}^{k} \binom{n}{i} < (k+1)\binom{n}{k}.$$

These edges cut C into at most k intervals, belonging alternately to V_1 and to V_2.

For a fixed partition with $2n/3 \geq |V_1| = n_1 \geq |V_2| = n_2 \geq n/3$, the number of perfect matchings of $V(H)$ with at most k edges running between V_1 and V_2 is at most

$$\sum_{i=0}^{k} \binom{n_1}{i}\binom{n_2}{i} i! \frac{(n_1-i)!}{(\frac{n_1-i}{2})!2^{\frac{n_1-i}{2}}} \frac{(n_2-i)!}{(\frac{n_2-i}{2})!2^{\frac{n_2-i}{2}}}$$

$$> (k+1)\binom{2n/3}{k}\binom{n/3}{k}k!\frac{(2n/3-k)!}{(\frac{2n/3-k}{2})!2^{\frac{2n/3-k}{2}}}\frac{(n/3-k)!}{(\frac{n/3-k}{2})!2^{\frac{n/3-k}{2}}}.$$

Indeed, one can choose i points from V_1 and i points from V_2 in $\binom{n_1}{i}\binom{n_2}{i}$ different ways, and match them in $i!$ ways. The number of matchings on the remaining $n_j - i$ points of V_j is at most $\frac{(n_j-i)!}{(\frac{n_j-i}{2})!2^{\frac{n_j-i}{2}}}$; $j = 1, 2$. Thus, the probability that $b(H_n) \leq k$ is at most

$$(k+1)^2\binom{n}{k}\binom{2n/3}{k}\binom{n/3}{k}k!\frac{(2n/3-k)!}{(\frac{2n/3-k}{2})!2^{\frac{2n/3-k}{2}}}\frac{(n/3-k)!}{(\frac{n/3-k}{2})!2^{\frac{n/3-k}{2}}}\Big/\frac{n!}{(n/2)!2^{n/2}},$$

which tends to zero when $k = \lfloor n/20 \rfloor$ and $n \to \infty$. □

Proof of Theorem 2 : Let $P = \{p_1, \ldots, p_n\}$ be the set of vertices of a convex n-gon in the plane, listed in clockwise order. Let G be a planar graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$, and let $f : V(G) \to P$ be a randomly chosen bijection. Suppose that G has m pairwise independent edges, $v_1v_2, \ldots, v_{2m-1}v_{2m}$.

Let H_{2m} be a graph on the vertex set $V(H_{2m}) = \{f(v_1), \ldots, f(v_{2m})\}$, constructed as follows. If $f(v_{i_1}), \ldots, f(v_{i_{2m}})$ is the list the elements of $V(H_{2m})$ in clockwise order around P, then let

$$E(H_{2m}) = \{f(v_{i_j})f(v_{i_{j+1}}) \mid 1 \leq j \leq 2m\} \cup \{f(v_{2i-1})f(v_{2i}) \mid 1 \leq i \leq m\},$$

where $i_{2m+1} := i_1$. Clearly, H_{2m} is isomorphic to the graph described in Lemma 5. In particular, almost surely we have $b(H_{2m}) \geq m/10$.

Suppose now, in order to obtain a contradiction, that G has a planar embedding which maps v_i to $f(v_i)$, $1 \leq i \leq n$, and every edge is represented by a polygonal curve such that at most $m/20$ edges are represented by curves with at least $m/40^3$ bends. If, for some $1 \leq i \leq m$, $v_{2i-1}v_{2i}$ is represented by a curve with at least $m/40^3$ bends, then remove $f(v_{2i-1})f(v_{2i})$ from the graph H_{2m}. The bisection width of the resulting graph H'_{2m} almost surely satisfies

$$b(H'_{2m}) \geq b(H_{2m}) - m/20 \geq m/10 - m/20 = m/20.$$

Hence, applying Lemma 4 with $d_i \leq 3$ $(1 \leq i \leq m)$, we obtain that almost surely

$$c(H'_m) \geq \frac{1}{16}\left(\frac{b^2(H'_m)}{(1.58)^2} - 9m\right) \geq \frac{(m/20)^2}{40} - m = \frac{m^2}{40 \times 20^2} - m.$$

On the other hand, in the above planar embedding of G, all paths representing the edges of H'_m have fewer than $m/40^3$ bends. Adding the edges of the convex hull of $f(v_{i_1}), \ldots, f(v_{i_{2m}})$ to the collection of these paths, we obtain a planar representation of H'_{2m} with at most $m \times (m/40^3 + 1) \times 2 \leq 2m^2/40^3 + 2m$ crossings, because each line segment in a polygonal path can cross at most two edges of the convex hull of P. Thus,

$$\frac{m^2}{40 \times 20^2} - m \leq c(H'_m) \leq \frac{m^2}{40^2 \times 20} + 2m,$$

a contradiction for suitably large m. □

4 Remarks

This paper discusses the worst case complexity of constructing a polygonal planar embedding of a graph G of n vertices, $v_1, \ldots, v_n$, where each vertex v_i is mapped to a prespecified point p_i. The corresponding optimization problem is the following: given a planar graph G with vertex set $\{v_1, \ldots, v_n\}$ and a point set $\{p_1, \ldots, p_n\}$, construct an embedding of G which maps v_i to p_i, $1 \leq i \leq n$, using as few bends (in total) as possible. Bastert and Fekete [1] proved that this problem is NP-hard. Is there an approximation algorithm for this problem which gives a solution within a factor of the optimal one?

Bastert and Fekete actually proved that minimizing the total number of bends is NP-hard in the case when G is a perfect matching. However, the complexity of the problem is unknown if G is a simple path, or, more generally, any connected graph. The complexity is also unknown if the points $p_1, \ldots, p_n$ are required to be in convex position (vertices of a convex polygon.) Are these restrictions of the problem also NP-hard?

The $\Omega(n^2)$ worst case lower bound in Theorem 2 assumes that the points $p_1, \ldots, p_n$ are in convex position. What if the points are not in convex position, say, they form a $\sqrt{n} \times \sqrt{n}$ grid? Are there examples of assignments of vertices of a planar graph G to the vertices of such a grid such that any polygonal planar embedding of G requires a total of $\Omega(n^2)$ bends?

References

1. Bastert, O., and Fekete, S. Geometrische verdrahtungsprobleme. Technical Report 96.247, Angewandte Mathematik und Informatik, Universität zu Köln, Köln, Germany, 1996.
2. Gritzmann, P., Mohar, B., Pach, J., and Pollack, R. Embedding a planar triangulation with vertices at specified points. *American Mathematical Monthly 98* (1991), 165–166. (Solution to problem E3341).
3. Lawrencenko, S., Liu, X., and Liu, Y. An algorithm for constructing a rectilinear embedding of a given graph in the plane. In *Combinatorics and graph theory '95* (River Edge, NJ, 1995), vol. 1, World Sci. Publishing, pp. 205–217.
4. Leighton, F. *Complexity Issues in VLSI.* MIT Press, Cambridge, 1983.
5. Liu, Y. *Embeddability in Graphs.* Kluwer, Amsterdam, 1995.
6. Liu, Y., Morgana, A., and Simeone, B. A linear algorithm for 2-bend embeddings of planar graphs in the two-dimensional grid. *Discrete Appl. Math. 81* (1998), 69–91.
7. Pach, J., and Agarwal, P. *Combinatorial Geometry.* John Wiley and Sons, New York, 1995.
8. Pach, J., Shahrokhi, F., and Szegedy, M. Applications of the crossing number. *Algorithmica 16* (1996), 111–117. (Special issue on Graph Drawing, edited by G. Di Battista and R. Tamassia.
9. Souvaine, D., and Wenger, R. Constructing piecewise linear homeomorphisms. Technical Report 94–52, DIMACS, New Brunswick, New Jersey, 1994.
10. Tamassia, R. On embedding a graph in the grid with the minimum number of bends. *SIAM J. Comput. 16*, 3 (1987), 421–444.