Problems and Results on Geometric Patterns

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Abstract: Many interesting problems in combinatorial and computational geometry can be reformulated as questions about occurrences of certain patterns in finite point sets. We illustrate this framework by a few typical results and list a number of unsolved problems.

1. Introduction: Models and Problems

We discuss some extremal problems on repeated geometric patterns in finite point sets in Euclidean space. Throughout this paper, a *geometric pattern* is an equivalence class of point sets in d-dimensional space under some fixed geometrically defined equivalence relation. Given such an equivalence relation and the corresponding concept of patterns, one can ask several natural questions:

(1) *What is the maximum number of occurrences of a given pattern among all subsets of an n-point set?*

(2) *How does the answer to the previous question depend on the particular pattern?*

(3) *What is the minimum number of distinct k-element patterns determined by a set of n points?*

These questions make sense for many specific choices of the underlying set and the equivalence relation. Hence it is not surprising that several basic problems of combinatorial geometry can be studied in this framework [PaA95].

In the simplest and historically first examples, due to Erdős [Er46], the underlying set consists of point pairs in the plane and the defining equivalence relation is the isometry (congruence). That is, two point pairs, $\{p_1, p_2\}$ and $\{q_1, q_2\}$, determine the same pattern if and only if $|p_1 - p_2| = |q_1 - q_2|$. In this case, (1) becomes the well known *Unit Distance Problem*: What is the maximum number of unit distance pairs determined by n points in the plane? It follows by scaling that the answer does not depend on the particular

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distance (pattern). For most other equivalence relations, this is not the case: different patterns may have different maximal multiplicities. For $k = 2$, question (3) becomes the *Problem of Distinct Distances*: What is the minimum number of distinct distances that must occur among n points in the plane? In spite of many efforts, we have no satisfactory answers to these questions. The best known results are the following.

Theorem 1. [SpST84] *Let $f(n)$ denote the maximum number of times the same distance can be repeated among n points in the plane. We have*

$$ne^{\Omega\left(\frac{\log n}{\log \log n}\right)} \leq f(n) \leq O\left(n^{\frac{4}{3}}\right).$$

Theorem 2. [KaT04] *Let $g(n)$ denote the minimum number of distinct distances determined by n points in the plane. We have*

$$\Omega\left(n^{0.8641}\right) \leq g(n) \leq O\left(\frac{n}{\sqrt{\log n}}\right).$$

In Theorems 1 and 2, the lower and upper bounds, respectively, are conjectured to be asymptotically sharp. See more about these questions in Section 3.

Erdős and Purdy [ErP71, ErP77] initiated the investigation of the analogous problems with the difference that, instead of pairs, we consider *triples* of points, and call two of them *equivalent* if the corresponding triangles have the same angle, or area, or perimeter. This leads to questions about the maximum number of equal angles, or unit-area resp. unit-perimeter triangles, that can occur among n points in the plane, and to questions about the minimum number of distinct angles, triangle areas, and triangle perimeters, respectively. Erdős's Unit Distance Problem and his Problem of Distinct Distances has motivated a great deal of research in extremal graph theory. The questions of Erdős and Purdy mentioned above and, in general, problems (1), (2), and (3) for larger than two-element patterns, require the extension of graph theoretic methods to hypergraphs. This appears to be one of the most important trends in modern combinatorics.

Geometrically, it is most natural to define two sets to be *equivalent* if they are congruent or similar to, or translates, homothets or affine images of each other. This justifies the choice of the word "pattern" for the resulting equivalence classes. Indeed, the algorithmic aspects of these problems have also been studied in the context of geometric pattern matching [AkTT98], [Br00], [AgS02], [Br02]. A typical algorithmic question is the following.

(4) *Design an efficient algorithm for finding all occurrences of a given pattern in a set of n points.*

It is interesting to compare the equivalence classes that correspond to the same relation applied to patterns of different sizes. If A and A' are equivalent under congruence (or under some other group of transformations mentioned above), and a is a point in A, then there exists a point $a' \in A'$ such that $A \setminus \{a\}$ is equivalent to $A' \setminus \{a'\}$. On the other hand, if A is equivalent (congruent) to A' and A is large enough, then usually its possible extensions are also determined: for each a, there exist only a small number of distinct elements a' such that $A \cup \{a\}$ is equivalent to $A' \cup \{a'\}$. Therefore, in order to bound the number of occurrences of a large pattern, it is usually sufficient to study small pattern fragments.

We have mentioned above that one can rephrase many extremal problems in combinatorial geometry as questions of type (1) (so-called *Turán-type* questions). Similarly, many *Ramsey-type* geometric coloring problems can also be formulated in this general setting.

(5) *Is it possible to color space with k colors such that there is no monochromatic occurrence of a given pattern?*

For point pairs in the plane under congruence, we obtain the famous Hadwiger-Nelson problem [Ha61]: What is the smallest number of colors $\chi(\mathbb{R}^2)$ needed to color all points of the plane so that no two points at unit distance from each other get the same color?

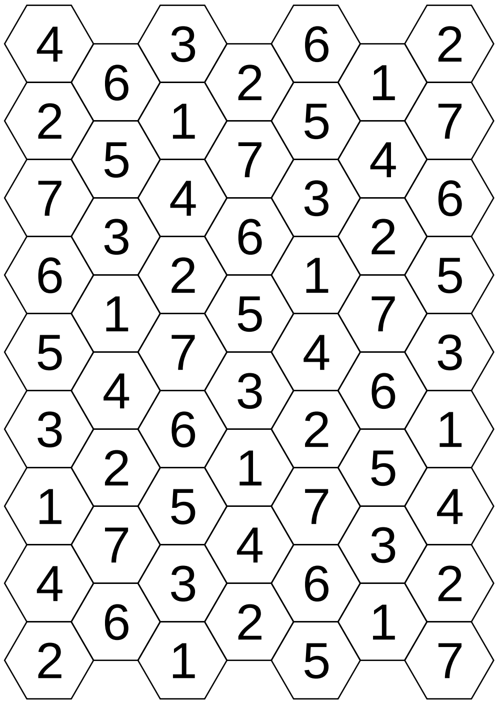

Fig. 1. *Seven coloring of the plane showing that* $\chi(\mathbb{R}^2) \leq 7$.

Theorem 3. $4 \leq \chi(\mathbb{R}^2) \leq 7$.

Another instance of question (5) is the following open problem from [ErG*73]: Is it possible to color all points of the three-dimensional Euclidean space with three colors so that no color class contains two vertices at distance one and the midpoint of the segment determined by them? It is known that four colors suffice, but there exists no such coloring with two colors. In fact, Erdős et al. [ErG*73] proved that for every d, the Euclidean d-space can be colored with four colors without creating a monochromatic triple of this kind.

2. A Simple Sample Problem: Equivalence under Translation

We illustrate our framework by analyzing the situation in the case in which two point sets are considered equivalent if and only if they are translates of each other. In this special case, we know the (almost) complete solution to problems (1)–(5) listed in the Introduction.

Theorem 4. *Any set B of n points in d-dimensional space has at most $n+1-k$ subsets that are translates of a fixed set A of k points. This bound is attained if and only if $A = \{p, p+v, \dots, p+(k-1)v\}$ and $B = \{q, q+v, \dots, q+(n-1)v\}$ for some $p, q, v \in \mathbb{R}^d$.*

The proof is simple. Notice first that no linear mapping φ that keeps all points of B distinct decreases the maximum number of translates: if $A+t \subset B$, then $\varphi(A)+\varphi(t) \subset \varphi(B)$. Thus, we can use any projection into $\mathbb{R}$, and the question reduces to the following one-dimensional problem: Given real numbers $a_1 < \dots < a_k$, $b_1 < \dots, b_n$, what is the maximum number of values t such that $t + \{a_1, \dots, a_k\} \subset \{b_1, \dots b_n\}$. Clearly, $a_1 + t$ must be one of $b_1, \dots, b_{n-k+1}$, so there are at most $n+1-k$ translates. If there are $n+1-k$ translates $t + \{a_1, \dots, a_k\}$ that occur in $\{b_1, \dots b_n\}$, for translation vectors $t_1 < \dots < t_{n-k+1}$, then $t_i = b_i - a_1 = b_{i+1} - a_2 = b_{i+j} - a_{1+j}$, for $i = 1, \dots, n-k+1$ and $j = 0, \dots, k-1$. But then $a_2 - a_1 = b_{i+1} - b_i = a_{j+1} - a_j = b_{i+j} - b_{i+j-1}$, so all differences between consecutive a_j and b_i are the same. For higher-dimensional sets, this holds for every one-dimensional projection, which guarantees the claimed structure. In other words, the maximum is attained only for sets of a very special type, which answers question (1).

An asymptotically tight answer to (2), describing the dependence on the particular pattern, was obtained in [Br03].

Theorem 5. *Let A be a set of points in d-dimensional space, such that the rational affine space spanned by A has dimension k. Then the maximum number of translates of A that can occur among n points in d-dimensional space is $n - \Theta(n^{\frac{k-1}{k}})$.*

Any set of the form $\{p, p+v, \dots, p+(k-1)v\}$ spans a one-dimensional rational affine space. An example of a set spanning a two-dimensional rational affine space is $\{0, 1, \sqrt{2}\}$, so for this set there are at most $n - \Theta(n^{1/2})$ possible translates. This bound is attained, e.g., for the set $\{i + j\sqrt{2} \mid 1 \leq i, j \leq \sqrt{n}\}$.

In this case, it is also easy to answer question (3), i.e., to determine the minimum number of distinct patterns (translation-inequivalent subsets) determined by an n-element set.

Theorem 6. *Any set of n points in d-dimensional space has at least $\binom{n-1}{k-1}$ distinct k-element subsets, no two of which are translates of each other. This bound is attained only for sets of the form $\{p, p+v, \ldots, p+(n-1)v\}$ for some $p, v \in \mathbb{R}^d$.*

By projection, it is again sufficient to prove the result on the line. Let $f(n,k)$ denote the minimum number of translation inequivalent k-element subsets of a set of n real numbers. Considering the set $\{1, \ldots, n\}$, we obtain that $f(n,k) \leq \binom{n-1}{k-1}$, since every equivalence class has a unique member that contains 1. To establish the lower bound, observe that, for any set of n real numbers, there are $\binom{n-2}{k-2}$ distinct subsets that contain both the smallest and the largest numbers, and none of them is translation equivalent to any other. On the other hand, there are at least $f(n-1,k)$ translation inequivalent subsets that do not contain the last element. So we have $f(n,k) \geq f(n-1,k) + \binom{n-2}{k-2}$, which, together with $f(n,1) = 1$, proves the claimed formula. To verify the structure of the extremal set, observe that, in the one-dimensional case, an extremal set minus its first element, as well as the same set minus its last element, must again be extremal sets, and for $n = k+1$ it follows from Theorem 1 that all extremal sets must form arithmetic progressions. Thus, the whole set must be an arithmetic progression, which holds, in higher-dimensional cases, for each one-dimensional projection.

The corresponding algorithmic problem (4) has a natural solution: Given two sets, $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_n\}$, we can fix any element of A, say, a_1, and try all possible image points b_i. Each of them specifies a unique translation $t = b_i - a_1$, so we simply have to test for each set $A + (b_i - a_1)$ whether it is a subset of B. This takes $\Theta(kn \log n)$ time. The running time of this algorithm is not known to be optimal.

Problem 1. *Does there exist an $o(kn)$-time algorithm for finding all real numbers t such that $t + A \subset B$, for every pair of input sets A and B consisting of k and n reals, respectively?*

The Ramsey-type problem (5) is trivial for translates. Given any set A of at least two points $a_1, a_2 \in A$, we can two-color $\mathbb{R}^d$ without generating any monochromatic translate of A. Indeed, the space can be partitioned into arithmetic progressions with difference $a_2 - a_1$, and each of them can be colored separately with alternating colors.

3. Equivalence under Congruence in the Plane

Problems (1)–(5) are much more interesting and difficult under congruence as the equivalence relation. In the plane, considering two-element subsets, the congruence class of a pair of points is determined by their distance. Questions (1) and (3) become the Erdős's famous problems, mentioned in the Introduction.

Problem 2. *What is the maximum number of times the same distance can occur among n points in the plane?*

Problem 3. *What is the minimum number of distinct distances determined by n points in the plane?*

The best known results concerning these questions were summarized in Theorems 1 and 2, respectively. There are several different proofs known for the currently best upper bound in Theorem 1 (see [SpST84], [ClE*90], [PaA95], and [Sz97]), which obviously does not depend on the particular distance (congruence class). This answers question (2). As for the lower bound of Katz and Tardos [KaT04] in Theorem 2, it represents the latest improvement over a series of previous results [SoT01], [Sz97], [ChST92], [Ch84], [Be83], [Mo52].

The algorithmic problem (4) can now be stated as follows.

Problem 4. *How fast can we find all unit distance pairs among n points in the plane?*

Some of the methods developed to establish the $O(n^{\frac{4}{3}})$ bound for the number of unit distances can also be used to design an algorithm for finding all unit distance pairs in time $O(n^{\frac{4}{3}} \log n)$ (similar to the algorithms for detecting point-line incidences [Ma93]).

The corresponding Ramsey-type problem (5) for patterns of size two is the famous Hadwiger-Nelson problem; see Theorem 3 above.

Problem 5. *What is the minimum number of colors necessary to color all points of the plane so that no pair of points at unit distance receive the same color?*

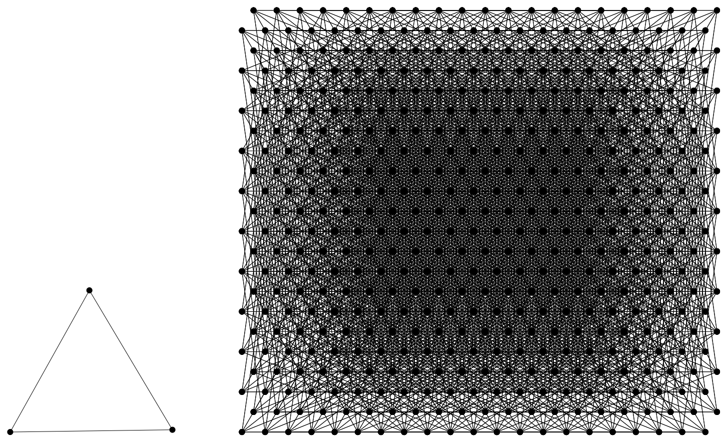

Fig. 2. *A unit equilateral triangle and a lattice section containing many congruent copies of the triangle*

If we ask the same questions for patterns of size k rather than point pairs, but still in the plane, the answer to (1) does not change. Given a pattern $A = \{a_1, \dots, a_k\}$, any congruent image of A is already determined, up to reflection, by the images of a_1 and a_2. Thus, the maximum number of congruent copies of a set is at most twice the maximum number of (ordered) unit distance pairs. Depending on the given set, this maximum number may be smaller, but no results of this kind are known. As n tends to infinity, the square and triangular lattice constructions that realize $ne^{\frac{c \log n}{\log \log n}}$ unit distances among n points also contain roughly the same number of congruent copies of *any* fixed set that is a subset of a square or triangular lattice. However, it is likely that this asymptotics cannot be attained for most other patterns.

Problem 6. *Does there exist, for every finite set A, a positive constant $c(A)$ with the following property: For every n, there is a set of n points in the plane containing at least $ne^{\frac{c(A) \log n}{\log \log n}}$ congruent copies of A ?*

The answer is yes if $|A| = 3$.

Problem (3) on the minimum number of distinct congruence classes of k-element subsets of a point set is strongly related to the Problem of Distinct Distances, just like the maximum number of pairwise congruent subsets was related to the Unit Distance Problem. For if we consider ordered k-tuples instead of k-subsets (counting each subset $k!$ times), then two such k-tuples are certainly incongruent if their first two points determine distinct distances. For each distance s, fix a point pair that determines s. Clearly, any two different extensions of a point pair by filling the remaining $k-2$ positions result in incongruent k-tuples. This leads to a lower bound of $\Omega(n^{k-2+0.8641})$ for the minimum number of distinct congruence classes of k-element subsets. Since a regular n-gon has $O(n^{k-1})$ pairwise incongruent k-element sets, this problem becomes less interesting for large k.

The algorithmic question (4) can also be reduced to the corresponding problem on unit distances. Given the sets A and B, we first fix $a_1, a_2 \in A$ and use our algorithm developed for detecting unit distance pairs to find all possible image pairs $b_1, b_2 \in B$ whose distance is the same as that of a_1 and a_2. Then we check for each of these pairs whether the rigid motion that takes a_i to b_i $(i = 1, 2)$ maps the whole set A into a subset of B. This takes $O^*(n^{4/3}k)$ time, and we cannot expect any substantial improvement in the dependence on n, unless we apply a faster algorithm for finding unit distance pairs. (In what follows, we write O^* to indicate that we ignore some lower order factors, i.e., $O^*(n^\alpha) = O(n^{\alpha+\varepsilon})$ for every $\varepsilon > 0$).

Many problems of Euclidean Ramsey theory can be interpreted as special cases of question (5) in our model. We particularly like the following problem raised in [ErG*75].

Problem 7. *Is it true that, for any triple $A = \{a_1, a_2, a_3\} \subset \mathbb{R}^2$ that does not span an equilateral triangle, and for any coloring of the plane with two colors, one can always find a monochromatic congruent copy of A?*

It was conjectured in [ErG*75] that the answer to this question is yes. It is easy to see that the statement is not true for equilateral triangles A. Indeed, decompose the plane into half-open parallel strips whose widths are equal to the height of A, and color them red and blue, alternately. On the other hand, the seven-coloring of the plane, with no two points at unit distance whose colors are the same, shows that any given pattern can be avoided with seven colors. Nothing is known about coloring with three colors.

Problem 8. *Does there exist a triple $A = \{a_1, a_2, a_3\} \subset \mathbb{R}^2$ such that any three-coloring of the plane contains a monochromatic congruent copy of A?*

4. Equivalence under Congruence in Higher Dimensions

All questions discussed in the previous section can also be asked in higher dimensions. There are two notable differences. In the plane, the image of a fixed pair of points was sufficient to specify a congruence. Therefore, the number of congruent copies of any larger set was bounded from above by the number of congruent pairs. In d-space, however, one has to specify d image points to determine a congruence, up to reflection. Hence, estimating the maximum number of congruent copies of a k-point set is a different problem for each $k = 2, \ldots, d$.

The second difference from the planar case is that starting from four dimensions, there exists another type of construction, discovered by Lenz, that provides asymptotically best answers to some of the above questions. For $k = \left\lfloor \frac{d}{2} \right\rfloor$, choose k concentric circles of radius $\frac{1}{\sqrt{2}}$ in pairwise orthogonal planes in $\mathbb{R}^d$ and distribute n points on them as equally as possible. Then any two points from distinct circles are at distance one, so the number of unit distance pairs is $(\frac{1}{2} - \frac{1}{2k} + o(1))n^2$, which is a positive fraction of all point pairs. It is known [Er60] that this constant of proportionality cannot be improved. Similarly, in this construction, any three points chosen from distinct circles span a unit equilateral triangle, so if $d \geq 6$, a positive fraction of all triples can be congruent. In general, for each $k \leq \left\lfloor \frac{1}{2}d \right\rfloor$, Lenz's construction shows that a positive fraction of all k-element subsets can be congruent. Obviously, this gives the correct order of magnitude for question (1). With some extra work, perhaps even the exact maxima can be determined, as has been shown for $k = 2, d = 4$ in [Br97], [vWa99].

Even for $k > \frac{d}{2}$, we do not know any construction better than Lenz's, but for these parameters the problem is not trivial. Now one is forced to pick several points from the same circle, and only one of them can be selected freely. So, for $d = 3$, in the interesting versions of (1), we have $k = 2$ or 3 (now there is no Lenz construction). For $d \geq 4$, the cases $\left\lfloor \frac{d}{2} \right\rfloor < k \leq d$ are nontrivial.

Problem 9. *What is the maximum number of unit distances among n points in three-dimensional space?*

Here, the currently best bounds are $\Omega(n^{\frac{4}{3}} \log\log n)$ [Er60] and $O^*(n^{\frac{3}{2}})$ [ClE*90].

Problem 10. *What is the maximum number of pairwise congruent triangles spanned by a set of n points in three-dimensional space?*

Here the currently best lower and upper bounds are $\Omega(n^{\frac{4}{3}})$ [ErHP89], [AbF02] and $O^*(n^{\frac{5}{3}})$ [AgS02], respectively. They improve previous results in [AkTT98], [Br00]. For higher dimensions, Lenz's construction or, in the odd-dimensional cases, a combination of Lenz's construction with the best known three-dimensional point set [ErHP89], [AbF02], are most likely to be optimal. The only results in this direction, given in [AgS02], are for $d \leq 7$ and do not quite attain this bound.

Problem 11. *Is it true that, for any $\left\lfloor \frac{d}{2} \right\rfloor \leq k \leq d$, the maximum number of congruent k-dimensional simplices among n points in d-dimensional space is $O(n^{\frac{d}{2}})$ if d is even, and $O(n^{\frac{d}{2}-\frac{1}{6}})$ if d is odd?*

Very little is known about problem (2) in this setting. For point pairs, scaling again shows that all two-element patterns can occur the same number of times. For three-element patterns (triangles), the aforementioned $\Omega(n^{\frac{4}{3}})$ lower bound in [ErHP89] was originally established only for right-angle isosceles triangles. It was later extended in [AbF02] to any fixed triangle. However, the problem is already open for full-dimensional simplices in 3-space. An especially interesting special case is the following.

Problem 12. *What is the maximum number of orthonormal bases that can be selected from n distinct unit vectors?*

The upper bound $O(n^{\frac{4}{3}})$ is simple, but the construction of [ErHP89] that gives $O(n^{\frac{4}{3}})$ orthogonal pairs does not extend to orthogonal triples.

Question (3) on the minimum number of distinct patterns is largely open. For two-element patterns, we obtain higher-dimensional versions of the Problem of Distinct Distances. Here the upper bound $O(n^{\frac{2}{d}})$ is realized, e.g., by a cubic section of the d-dimensional integer lattice. The general lower bound of $\Omega(n^{\frac{1}{d}})$ was observed already in [Er46]. For $d = 3$, this was subsequently improved to $\Omega^*(n^{\frac{77}{141}})$ [ArP*03] and to $\Omega(n^{0.564})$ [SoV05]. For large values of d, Solymosi and Vu [SoV05] got very close to finding the best exponent by establishing the lower bound $\Omega\left(n^{\frac{2}{d}-\frac{2}{d(d+2)}}\right)$. This extends, in the same way as in the planar case, to a bound of $\Omega\left(n^{k-2+\frac{2}{d}-\frac{2}{d(d+2)}}\right)$ for the minimum number of distinct k-point patterns of an n-element set, but even for triangles, nothing better is known. Lenz-type constructions are not useful in this context, because they span $\Omega(n^{k-1})$ distinct k-point patterns, as do regular n-gons.

As for the algorithmic problem (4), it is easy to find all congruent copies of a given k-point pattern A in an n-point set. For any $k \geq d$, this can be achieved in $O(n^d k \log n)$ time: fix a d-tuple C in A, and test all d-tuples of the n-point set B, whether they could be an image of C. If yes, test whether the congruence specified by them maps all the remaining $k-d$ points to elements of B. It is very likely that there are much faster algorithms, but, for general d, the only published improvement is by a factor of $\log n$ [dReL95].

The Ramsey-type question (5) includes a number of problems of Euclidean Ramsey theory, as special cases.

Problem 13. *Is it true that for every two-coloring of the three-dimensional space, there are four vertices of the same color that span a unit square?*

It is easy to see that if we divide the plane into half-open strips of width one and color them alternately by two colors, then no four vertices that span a unit square will receive the same color. On the other hand, it is known that any two-coloring of four-dimensional space will contain a monochromatic unit square [ErG*75]. Actually, the (vertex set of a) square is one of the simplest examples of a *Ramsey set*, i.e., a set B with the property that, for every positive integer c, there is a constant $d = d(c)$ such that under any c-coloring of the points of $\mathbb{R}^d$ there exists a monochromatic congruent copy of B. All boxes, all triangles [FrR86], and all trapezoids [Kř92] are known to be Ramsey. It is a long-standing open problem to decide whether all finite subsets of finite dimensional spheres are Ramsey. If the answer is in the affirmative, this would provide a perfect characterization of Ramsey sets, for all Ramsey sets are known to be subsets of a sphere [ErG*73].

The simplest nonspherical example, consisting of an equidistant sequence of three points along the same line, was mentioned at the end of the Introduction.

5. Equivalence under Similarity

If we consider problems (1)–(5) with similarity (congruence and scaling) as the equivalence relation, again we find that many of the resulting questions have been extensively studied. Since any two point pairs are similar to each other, we can restrict our attention to patterns of size at least three. The first interesting instance of problem (1) is to determine or to estimate the maximum number of pairwise similar triangles spanned by n points in the plane. This problem was almost completely solved in [ElE94]. For any given triangle, the maximum number of similar triples in a set of n point in the plane is $\Theta(n^2)$. If the triangle is equilateral, we even have fairly good bounds on the multiplicative constants hidden in the Θ-notation [AbF00]. In this case, most likely, suitable sections of the triangular lattice are close to being extremal for (1). In general, the following construction from [ElE94] always gives a quadratic number of similar copies of a given triangle $\{a,b,c\}$. Interpreting a,b,c as complex numbers $0,1,z$, consider the points $\frac{i_1}{n}z$, $\frac{i_2}{n}+(1-\frac{i_2}{n})z$, and $\frac{i_3}{n}z+(1-\frac{i_3}{n})z^2$, where $0 < i_1, i_2, i_3 \leq \frac{n}{3}$. Then any triangle $(\beta-\alpha)z$, $\alpha+(1-\alpha)z$, $\beta z+(1-\beta)z^2$ is similar to $0,1,z$, which can be checked by computing the ratios of the sides. Thus, choosing $\alpha = \frac{i_2}{n}$, $\beta = \frac{i_3}{n}$, we obtain a quadratic number of similar copies of the triangle $0, 1, z$.

The answer to question (1) for k-point patterns, $k > 3$, is more or less the same as for $k = 3$. Certain patterns, including all k-element subsets of a regular triangular lattice, permit $\Theta(n^2)$ similar copies, and in this case a suitable section of the triangular lattice is probably close to being extremal. For some other patterns, the order $\Theta(n^2)$ cannot be attained. All patterns of the former type were completely characterized in [LaR97]: for any pattern A of $k \geq 4$ points, one can find n points containing $\Theta(n^2)$ similar copies of A if and only if the cross ratio of every quadruple of points in A, interpreted as complex numbers, is algebraic. Otherwise, the maximum is slightly subquadratic. This result also answers question (2).

In higher dimensions, the situation is entirely different: we do not have good bounds for question (1) in any nontrivial case. The first open question is to determine the maximum number of triples in a set of n points in 3-space that induce pairwise similar triangles. The trivial upper bound, $O(n^3)$, was reduced to $O(n^{2.2})$ in [AkTT98]. On the other hand, we do not have any better lower bound than $\Omega(n^2)$, which is already valid in the plane. These estimates extend to similar copies of k-point patterns, $k > 3$, provided that they are planar.

Problem 14. *What is the maximum number of pairwise similar triangles induced by n points in three-dimensional space?*

For full-dimensional patterns, no useful constructions are known. The only lower bound we are aware of follows from the lattice L which, in three dimensions, spans $\Omega(n^{\frac{4}{3}})$ similar copies of the full-dimensional simplex formed by its basis vectors or, in fact, of any k-element subset of lattice points. However, to attain this bound, we do not need to allow rotations: L spans $\Omega(n^{\frac{4}{3}})$ *homothetic* copies.

Problem 15. *In three-dimensional space, what is the maximum number of quadruples in an n-point set that span pairwise similar tetrahedra?*

For higher dimensions and for larger pattern sizes, the best known lower bound follows from Lenz's construction for congruent copies, which again does not use the additional freedom of scaling. Since, for $d \geq 3$, we do not know the answer to question (1) on the maximum number occurrences, there is little hope that we would be able to answer question (2) on the dependence of this maximum number on the pattern.

Problem (3) on the minimum number of pairwise inequivalent patterns under similarity is an interesting problem even in the plane.

Problem 16. *What is the minimum number of similarity classes of triangles spanned by a set of n points in the plane?*

There is a trivial lower bound of $\Omega(n)$: if we choose two arbitrary points, and consider all of their $n-2$ possible extensions to a triangle, then among these triangles each (oriented) similarity class will be represented only at most three times. Alternatively, we obtain asymptotically the same lower bound $\Omega(n)$ by just using the pigeonhole principle and the fact that the maximum size of a similarity class of triangles is $O(n^2)$. On the other hand, as shown by the example of a regular n-gon, the number of similarity classes of triangles can be $O(n^2)$. This leaves a huge gap between the lower and upper bounds.

For higher dimensions and for larger sets, our knowledge is even more limited. In three-dimensional space, for instance, we do not even have an $\Omega(n)$ lower bound for the number of similarity classes of triangles, while the best known upper bound, $O(n^2)$, remains the same. For four-element patterns, we have a linear lower bound (fix any triangle, and consider its extensions), but we have no upper bound better than $O(n^3)$ (consider again a regular n-gon). Here we have to be careful with the precise statement of the problem. We have to decide whether we count similarity classes of *full-dimensional* simplices only, or all similarity classes of possibly degenerate four-tuples. A regular $(n-1)$-gon with an additional point on its axis has only $\Theta(n^2)$ similarity classes of full-dimensional simplices, but $\Theta(n^3)$ similarity classes of four-tuples. In dimensions larger than three, nothing nontrivial is known.

In the plane, the algorithmic question (4) of finding all similar copies of a fixed k-point pattern is not hard: it can be achieved in time $O(n^2 k \log n)$, which is tight up to the $\log n$-factor, because the output complexity can be as large as $\Omega(n^2 k)$ in the worst case. For dimensions three and higher, we have no nontrivial algorithmic results. Obviously, the problem can always be solved in $O(n^d k \log n)$ time, by testing all possible d-tuples of the underlying set, but this is probably far from optimal.

The Ramsey-type question (5) has a negative answer, for any finite number of colors, even for homothetic copies. Indeed, for any finite set A and for any coloring of space with a finite number of colors, one can always find a monochromatic set similar (even homothetic) to A. This follows from the Hales–Jewett theorem [HaJ63], which implies that every coloring of the integer lattice $\mathbf{Z}^d$ with a finite number of colors contains a monochromatic homothetic copy of the lattice cube $\{1, \ldots, m\}^d$ (Gallai–Witt theorem [Ra43], [Wi52]).

6. Equivalence under Homothety or Affine Transformations

For homothety-equivalence, questions (1) and (2) have been completely answered in all dimensions [vKrB89], [ElE94], [Br02]. The maximum number of homothetic copies of a set that can occur among n points is $\Theta(n^2)$; the upper bound $O(n^2)$ is always trivial, since the image of a set under a homothety is specified by the images of two points; and a lower bound of $\Omega(n^2)$ is attained by the homothetic copies of $\{1, \ldots, k\}$ in $\{1, \ldots, n\}$. The maximum order is attained only for this one-dimensional example. If the dimension of the affine space induced by a given pattern A over the rationals is k, then the maximum number of homothetic copies of A that can occur among n points is $\Theta(n^{1+\frac{1}{k}})$, which answers question (2).

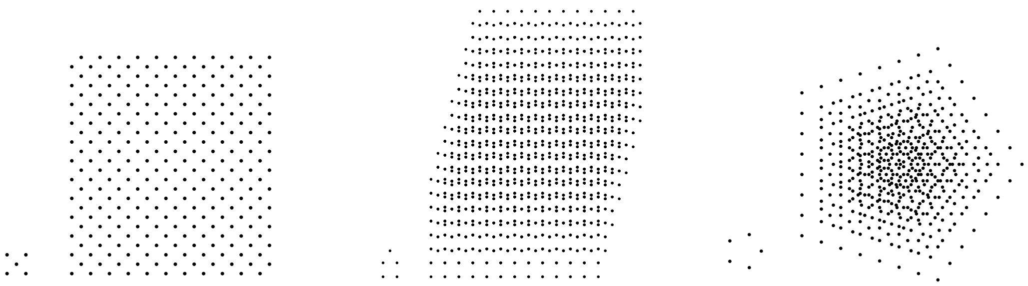

Fig. 3. *Three five-point patterns of different rational dimensions and three sets containing many of their translates*

Question (3) on the minimum number of distinct homothety classes of k-point subsets among n points, seems to be still open. As in the case of translations, by projection, we can restrict our attention to the one-dimensional case, where a sequence of equidistant points $\{0, \ldots, n-1\}$ should be extremal. This gives $\Theta(n^{k-1})$ distinct homothety classes. To see this, notice that as the size of the sequence increases from $n-1$ to n, the number of additional homothety classes that were not already present in $\{0, \ldots n-2\}$, is $\Theta(n^{k-2})$. (The increment certainly includes the classes of all k-tuples that contain 0, $n-1$, and a third number coprime to $n-1$.) Unfortunately, the pigeonhole principle gives only an $\Omega(n^{k-2})$ lower bound for the number of pairwise dissimilar k-point patterns spanned by a set of n numbers.

Problem 17. *What is the minimum number of distinct homothety classes among all k-element subsets of a set of n numbers?*

The algorithmic problem (4) was settled in [vKrB89], [Br02]. All homothetic copies of a given full-dimensional k-point pattern in an n-element set in d-space can be found in $O(n^{1+\frac{1}{d}} k \log n)$ time. This is asymptotically tight up to the $\log n$-factor. As mentioned in the previous section, the answer to the corresponding Ramsey-type question (5), is negative: one cannot avoid monochromatic homothetic copies of any finite pattern with any finite number of colors.

The situation is very similar for affine images. The maximum number of affine copies of a set among n points in d-dimensional space is $\Theta(n^{d+1})$. The upper bound is trivial, since an affine image is specified by the images of $d+1$ points. On the other hand, the d-dimensional "lattice cube," $\{1, \ldots, n^{\frac{1}{d}}\}^d$, contains $\Omega(n^{d+1})$ affine images of $\{0,1\}^d$ or of any other small lattice-cube of fixed size.

The answer to question (2) is not so clear.

Problem 18. *Do there exist, for every full-dimensional pattern A in d-space, n-element sets containing $\Omega(n^{d+1})$ affine copies of A?*

Problem 19. *What is the minimum number of affine equivalence classes among all k-element subsets of a set of n points in d-dimensional space?*

For the algorithmic problem (4), the brute force method of trying all possible $(d+1)$-tuples of image points is already optimal. The Ramsey-type question (5) has again a negative answer, since every homothetic copy is also an affine copy.

7. Other Equivalence Relations for Triangles in the Plane

For triples in the plane, several other equivalence relations have been studied. An especially interesting example is the following. Two ordered triples are considered equivalent if they determine the same angle. It was proved in [PaS92] that the maximum number of triples in a set of n points in the plane that determine the same angle α is $\Theta(n^2 \log n)$. This order of magnitude is attained for a dense set of angles α. For every other angle α, distribute as evenly as possible $n-1$ points on two rays that emanate from the origin and enclose angle α, and place the last point at the origin. Clearly, the number of triples determining angle α is $\Omega(n^2)$, which "almost" answers question (2). As for the minimum number of distinct angles determined by n points in the plane, Erdős conjectured that the answer to the following question is in the affirmative.

Problem 20. *Is it true that every set of n points in the plane, not all on a line, determine at least $n-2$ distinct angles?*

This number is attained for a regular n-gon and for several other configurations.

The corresponding algorithmic question (4) is easy: list, for each point p of the set, all lines ℓ through p, together with the points on ℓ. Then we can find all occurrences of a given angle in time $O(n^2 \log n + a)$, where a is the number of occurrences of that angle. Thus, by the above bound from [PaS92], the problem can be solved in $O(n^2 \log n)$ time, which is optimal. The negative answer to the Ramsey-type question (5) again follows from the analogous result for homothetic copies: no coloring with a finite number of colors can avoid a given angle.

Another natural equivalence relation classifies triangles according to their areas.

Problem 21. *What is the maximum number of unit-area triangles that can be determined by n points in the plane?*

An upper bound of $O(n^{\frac{7}{3}})$ was established in [PaS92], while it was pointed out in [ErP71] that a section of the integer lattice gives the lower bound $\Omega(n^2 \log \log n)$. By scaling, we see that all areas allow the same multiplicities, which answers (2). However, problem (3) is open in this case.

Problem 22. *Is it true that every set of n points in the plane, not all on a line, spans at least $\left\lfloor \frac{n-1}{2} \right\rfloor$ triangles of pairwise different areas?*

This bound is attained by placing on two parallel lines two equidistant point sets whose sizes differ by at most one. This construction is conjectured to be extremal [ErP77], [St78]. The best known lower bound, $0.4142n - O(1)$, follows from [BuP79], using [Un82].

The corresponding algorithmic problem (4) is to find all unit-area triangles. Again, this can be done in $O(n^2 \log n + a)$ time, where a denotes the number of unit area triangles. First, dualize the points to lines, and construct their arrangement, together with a point location structure. Next, for each pair (p, q) of original points, consider the two parallel lines that contain all points r such that pqr is a triangle of unit area. These lines correspond to points in the dual arrangement, for which we can perform a point location query to determine all dual lines containing them. They correspond to points in the original set that together with p and q span a triangle of area one. Each such query takes $\log n$ time plus the number of answers returned.

Concerning the Ramsey-type problem (4), it is easy to see that, for any 2-coloring of the plane, there is a monochromatic triple that spans a triangle of unit area. The same statement may hold for any coloring with a finite number of colors.

Problem 23. *Is it true that for any coloring of the plane with a finite number of colors, there is a monochromatic triple that spans a triangle of unit area?*

The *perimeter* of triangles was also discussed in the same paper [PaS92], and later in [PaS04], where an upper bound of $O(n^{\frac{16}{7}})$ was established, but there is no nontrivial lower bound. The lattice section has $\Omega(ne^{\frac{c \log n}{\log \log n}})$ pairwise *congruent* triangles, which, of course, also have equal perimeters, but this bound is probably far from being sharp.

Problem 24. *What is the maximum number of unit perimeter triangles spanned by n points in the plane?*

By scaling, all perimeters are equivalent, answering (2). By the pigeonhole principle, we obtain an $\Omega(n^{\frac{5}{7}})$ lower bound for the number of distinct perimeters, but again this is probably far from the truth.

Problem 25. *What is the minimum number of distinct perimeters assumed by all $\binom{n}{3}$ triangles spanned by a set of n points in the plane?*

Here neither the algorithmic problem (4) nor the Ramsey-type problem (5) has an obvious solution. Concerning the latter question, it is clear that with a sufficiently large number of colors, one can avoid unit perimeter triangles: color the plane "cellwise," where each cell is too small to contain a unit perimeter triangle, and two cells of the same color are far apart. The problem of determining the minimum number of colors required seems to be similar to the question addressed by Theorem 3.

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