Crossing patterns of segments

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Dedicated to the memory of Paul Turán on his 90*th birthday*

Abstract

It is shown that for every $c > 0$ there exists $c' > 0$ satisfying the following condition. Let $\mathcal{S}$ be a system of n straight-line segments in the plane, which determine at least cn^2 crossings. Then there are two disjoint at least $c'n$-element subsystems, $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$, such that every element of $\mathcal{S}_1$ crosses all elements of $\mathcal{S}_2$.

1 Introduction, results

Given a system $\mathcal{S}$ of simple continuous curves ("strings") in the plane, we can define a graph $G_{\mathcal{S}}$ as follows. Assign a vertex to each curve, and connect two vertices by an edge if and only if the corresponding two curves intersect. $G_{\mathcal{S}}$ is called the *intersection graph* of $\mathcal{S}$.

Not every graph is an intersection graph of a system of curves [EET76] (see Figure 1 for a simple example). This implies that only a very small fraction of all the $2^{\binom{n}{2}}$ labeled graphs on n vertices have this property. For

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systems of *segments*, using standard techniques from real algebraic geometry (see [GP86],[AKS90],[PPW90]), we obtain a fairly good quantitative result.

Theorem 1. *The number of labeled graphs on n vertices which can be obtained as the intersection graph of a system of segments in the plane is $2^{O(n \log n)}$.*

The problem of recognizing intersection graphs of planar curves (the so-called "string graph problem") is known to be NP-hard [K91], but it is open whether this problem is *decidable* [KM91]. In some very special cases, e.g., when $\mathcal{S}$ consists of segments, there are trivial recognition algorithms [CGP98], [FMP95]. But even in these cases we do not know much about the structure of intersection graphs. One of the most striking examples illustrating our ignorance in the subject is the following simple open

Problem. *Is it true that every planar graph is the intersection graph of a system of segments in the plane?*

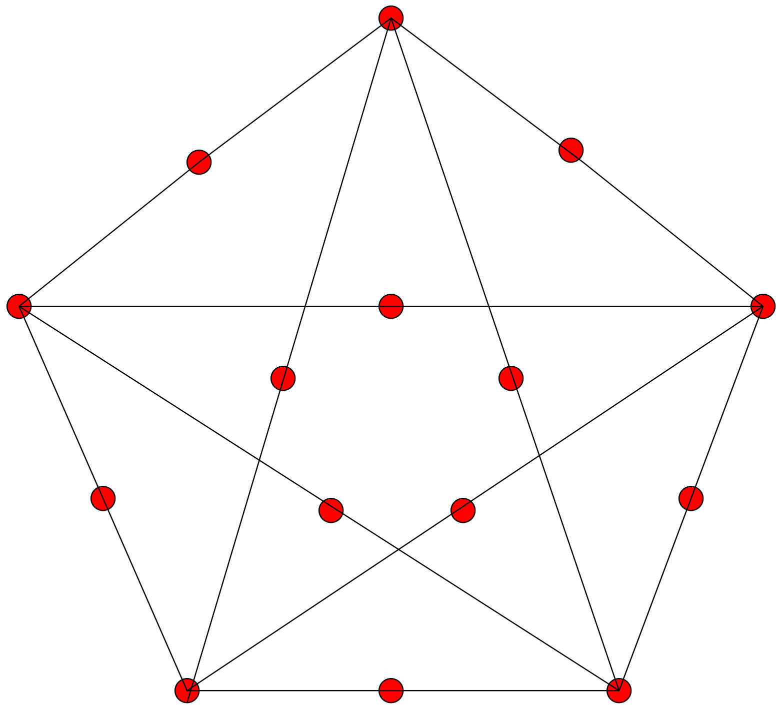

Figure 1. A graph which is *not* the intersection graph of any system of segments

The aim of this paper is to prove some Ramsey-type results for intersection graphs of segments. In other words, we establish *necessary* conditions for a graph to be the intersection graph of a system of segments. We recall a simple observation from Ramsey theory. As usual, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively.

Theorem. [EHP00] *Let H be a fixed graph of k vertices. Then every graph G with n vertices, which does not contain an induced subgraph isomorphic to H, has two disjoint sets of vertices, $V_1, V_2 \subset V(G)$, such that $|V_1|, |V_2| \geq n^{1/(k-1)}/2$ and*

(i) either all edges between V_1 and V_2 belong to G,

(ii) or no edge between V_1 and V_2 belongs to G. $\Box$

Note that the weaker result, with roughly $\log n$ in the place of $n^{1/(k-1)}$, immediately follows from Ramsey's theorem [ES35]. Combining the last theorem with Theorem 1 (or rather with the fact that there is *at least one* forbidden induced subgraph in the class of all segment intersection graphs, say, the 15 vertex graph depicted in Figure 1), we obtain the following

Corollary. *There exists a constant $\varepsilon \geq 1/14$ such that every system $\mathcal{S}$ of n segments in the plane has two disjoint subsystems $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that $|\mathcal{S}_1|, |\mathcal{S}_2| \geq n^{\varepsilon}/2$ and*

(i) either every segment in $\mathcal{S}_1$ crosses all segments in $\mathcal{S}_2$,

(ii) or no segment in $\mathcal{S}_1$ crosses any segment in $\mathcal{S}_2$.

Note that here, as well as in the sequel, we only consider systems of segments in *general position*, i.e., we assume that no two segments are parallel and no three endpoints are collinear.

The main result of this paper, formulated in the next two statements, substantially strengthens the last Corollary. In all of these results, A stands for an absolute constant smaller than 10^6.

Theorem 2. *Any system $\mathcal{S}$ of n segments in the plane with at least cn^2 crossings ($c > 0$) has two disjoint subsystems, $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$, such that $|\mathcal{S}_1|, |\mathcal{S}_2| \geq \frac{(2c)^A}{660} n$ and every segment in $\mathcal{S}_1$ crosses all segments in $\mathcal{S}_2$.*

Theorem 3. *Any system $\mathcal{S}$ of n segments in the plane with at least cn^2 non-crossing pairs ($c > 0$) has two disjoint subsystems, $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$, such that $|\mathcal{S}_1|, |\mathcal{S}_2| \geq \frac{(c/5)^A}{330} n$ and no segment in $\mathcal{S}_1$ crosses any segment in $\mathcal{S}_2$.*

A *geometric graph* is a graph whose vertices are points in general position in the plane (i.e., no three points are on a line) and whose edges are straight-line segments connecting these points. Our last two results are easy corollaries to Theorems 2 and 3, respectively.

Theorem 4. *Any geometric graph G with n vertices and at least cn^2 edges ($c > 0$) has two disjoint sets of edges $E_1, E_2 \subset E(G)$ such that $|E_1|, |E_2| \geq (c/32)^{A+3}\binom{n}{2}$ and every edge in E_1 crosses all edges in E_2.*

Theorem 5. *Any geometric graph G with n vertices and at least cn^2 edges ($c > 0$) has two disjoint sets of edges $E_1, E_2 \subset E(G)$ such that $|E_1|, |E_2| \geq (c/34)^{A+3}\binom{n}{2}$ and no edge in E_1 crosses any edge in E_2.*

The rest of the paper is organized as follows. In Section 2, we establish Theorem 1. Theorems 2 and 3 are proved in Section 3. The last section contains the proofs of Theorems 4 and 5, as well as some concluding remarks.

2 Bounding the number of intersection graphs of segments

The aim of this section is to prove Theorem 1.

Let $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ be a system of segments in general position in the plane. Assume that s_i is not parallel to the y-axis and can be described by the relations:

$$s_i: \quad y = a_i x + b_i, \; c_i \leq x \leq d_i \quad (i = 1, 2, \ldots, n).$$

Two segments, s_i and s_j, cross each other if and only if

$$\max\{c_i, c_j\} \leq \frac{b_j - b_i}{a_i - a_j} \leq \min\{d_i, d_j\}.$$

Thus, whether or not s_i crosses s_j, is determined by the sign of polynomials

$$P_{\{i,j\}} := a_i - a_j, \quad Q_{\{i,j\}} := c_i(a_i - a_j) + b_i - b_j,$$

for any distinct $i \neq j \in \{1, 2, \ldots, n\}$. These are $2\binom{n}{2}$ polynomials of degree at most 2 in the $4n$ variables $\{a_i, b_i, c_i, d_i\}$.

We use the following form of the Milnor-Thom theorem [M64],[T65].

Theorem. [W68] *The number of different sign patterns for m polynomials of degree at most d in k variables is at most $(4edm/k)^k$.*

Applying this bound with $m = 2\binom{n}{2}$, $d = 2$, and $k = 4n$, we obtain that there are at most $(2e(n-1))^{4n} = 2^{O(n \log n)}$ different ways how the crossing relationship can be defined among n segments in the plane.

3 Proofs of Theorems 2 and 3

Three sets of points in the plane are said to be *separable* if each of them can be separated from the other two by a straight line. Given three separable sets, there is no straight line which intersects the convex hull of all of them.

Lemma 3.1. [BE00] *Every set of n points in general position in the plane has three separable subsets of size $\lfloor n/6 \rfloor$.*

Proof: Assume without loss of generality that n is divisible by 6, and let P be an n-element point set. Choose two lines that divide the plane into 4 regions, containing $n, 2n, n$, and $2n$ points of P in their interiors, in this cyclic order. Let P_1, P_2, P_3, and P_4 denote the corresponding subsets of P. By the *ham-sandwich theorem*, there is a line ℓ which simultaneously cuts P_2 and P_4 into two halves of equal size (see Figure 2). Then ℓ avoids either the convex hull of P_1 or that of P_3. Assume, by symmetry, that P_1 is 'above' ℓ. Then P_1 and the parts of P_2 and P_4 'below' ℓ are three separable sets. (Similar arguments can be found e.g. in [BV98].) □

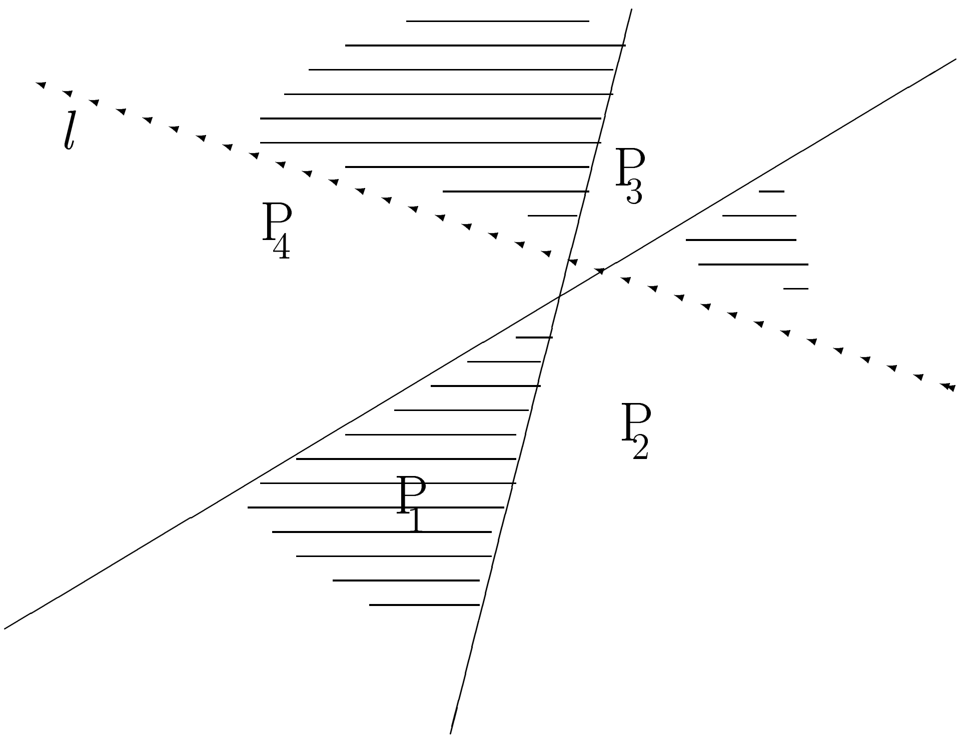

Figure 2.

Lemma 3.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be two systems of segments in general position in the plane. Then there are two subsystems $\mathcal{S}^* \subseteq \mathcal{S}$, $\mathcal{T}^* \subseteq \mathcal{T}$ such that $|\mathcal{S}^*| \geq \lfloor|\mathcal{S}|/330\rfloor$, $|\mathcal{T}^*| \geq \lfloor|\mathcal{T}|/330\rfloor$, and*

(i) either every segment in $\mathcal{S}^*$ crosses all segments in $\mathcal{T}^*$,

(ii) or no segment in $\mathcal{S}^*$ crosses any segment in $\mathcal{T}^*$.

Proof: Let $|\mathcal{S}| = m, |\mathcal{T}| = n$, and suppose, for simplicity, that both m and n are multiples of 330. Let P be the set of endpoints of all segments in $\mathcal{S}$. By Lemma 3.1, there are three separable $m/3$-element subsets, $P_1, P_2, P_3 \subseteq P$. Color a segment $t \in \mathcal{T}$ with color i if its supporting line does not intersect the convex hull of P_i $(i = 1, 2, 3)$. Let $\mathcal{T}_i$ denote the segments of color i. At least one third of the elements of $\mathcal{T}$ get the same color, so we can assume with no loss of generality that $|\mathcal{T}_1| \geq n/3$.

If there are at least $m/330$ segments in $\mathcal{S}$, both of whose endpoints belong to P_1, then we are done, because these segments are disjoint from all elements of $\mathcal{T}_1$.

Hence, we can assume that at least $(1/3 - 2/330)m = 18m/55$ elements of $\mathcal{S}$ have precisely one of their endpoints in P_1. Let Q denote the set of other endpoints of these segments. Let us choose three separable subsets $Q_1, Q_2, Q_3 \subseteq Q$, each of size at least $|Q|/6 = 3m/55$. Just as before, color a segment $t \in \mathcal{T}_1$ with color i if its supporting line does not intersect the convex hull of Q_i $(i = 1, 2, 3)$. Again, at least $|\mathcal{T}_1|/3 \geq n/9$ elements of $\mathcal{T}_1$ get the same color, say color 1; they form a subsystem $\mathcal{T}_{11} \subseteq \mathcal{T}_1$.

Let $\mathcal{S}_{11}$ denote set of all elements of $\mathcal{S}$ with one endpoint in P_1 and the other in Q_1. Clearly, we have $|\mathcal{S}_{11}| = |Q_1| \geq 3m/55$.

Let us repeat now the whole procedure with $\mathcal{T}_{11}$ in the place of $\mathcal{S}$ and $\mathcal{S}_{11}$ in the place of $\mathcal{T}$. We obtain two subsets, $\mathcal{T}' \subseteq \mathcal{T}_{11}$ and $\mathcal{S}' \subseteq \mathcal{S}_{11}$, satisfying

$$|\mathcal{T}'| \geq \frac{3|\mathcal{T}_{11}|}{55} \geq \frac{n}{165}, \quad |\mathcal{S}'| \geq \frac{|\mathcal{S}_{11}|}{9} \geq \frac{m}{165}.$$

We can assume that at least half of the supporting lines of the elements of $\mathcal{T}'$ cross the convex hull of $\mathcal{S}'$, for otherwise we would obtain two non-crossing systems of at least $|\mathcal{T}'|/2$ and $|\mathcal{S}'|$ segments. The set of of all elements of $\mathcal{T}'$, whose supporting lines cross the convex hull of $\mathcal{S}'$ is denoted by $\mathcal{T}^*$. Similarly, we can assume that the supporting lines of at least half of the elements of $\mathcal{S}'$ cross the convex hull of $\mathcal{T}^*$; otherwise, we could find two non-crossing systems of at least $|\mathcal{T}^*|$ and $|\mathcal{S}'|/2$ segments. Let $\mathcal{S}^*$ denote the set of all elements of

$\mathcal{S}'$, whose supporting lines cross the convex hull of $\mathcal{T}^*$. It follows from the definitions that every element of $\mathcal{S}^*$ crosses all elements of $\mathcal{T}^*$ and that

$$|\mathcal{S}^*| \geq \frac{|\mathcal{S}|'}{2} \geq \frac{m}{330}, \quad |\mathcal{T}^*| \geq \frac{|\mathcal{T}|'}{2} \geq \frac{n}{330}. \quad \square$$

Given any two systems of segments, $\mathcal{S}$ and $\mathcal{T}$, in general position in the plane, define their *crossing density*, $\delta(\mathcal{S},\mathcal{T})$, as the number of crossing pairs $(s,t),\ s \in \mathcal{S},\ t \in \mathcal{T}$ divided by $|\mathcal{S}| \cdot |\mathcal{T}|$. Clearly, we have $0 \leq \delta(\mathcal{S},\mathcal{T}) \leq 1$.

Theorems 2 and 3 readily follow from the next result.

Theorem 3.3. *There exists a constant $A < 10^6$ satisfying the following condition. Let $\mathcal{S}$ and $\mathcal{T}$ be any sets of segments in general position in the plane, and suppose that their crossing density is at least $c > 0$. Then there are two disjoint subsystems $\mathcal{S}' \subseteq \mathcal{S}$, $\mathcal{T}' \subseteq \mathcal{T}$ such that*

$$|\mathcal{S}'| \geq \frac{c^A}{330}|\mathcal{S}|, \quad |\mathcal{T}'| \geq \frac{c^A}{330}|\mathcal{T}|,$$

and every segment in $\mathcal{S}'$ crosses all segments in $\mathcal{T}'$.

Proof: Let $|\mathcal{S}| = m, |\mathcal{T}| = n$, and suppose first that both m and n are powers of 330. According to our assumption, $\delta(\mathcal{S},\mathcal{T}) \geq c$.

Applying Lemma 3.2, we obtain two subsystems, $\mathcal{S}^* \subset \mathcal{S}$, $\mathcal{T}^* \subset \mathcal{T}$, such that $|\mathcal{S}^*| = m/330, |\mathcal{T}^*| = n/330$, and $\delta(\mathcal{S}^*,\mathcal{T}^*)$ is either 1 or 0. In the first case we are done, so assume $\delta(\mathcal{S}^*,\mathcal{T}^*) = 0$. Then we have

$$c \leq \delta(\mathcal{S},\mathcal{T}) = \frac{329}{330^2}\delta(\mathcal{S},\mathcal{T}-\mathcal{T}^*) + \frac{329}{330^2}\delta(\mathcal{S}-\mathcal{S}^*,\mathcal{T}) + \frac{329^2}{330^2}\delta(\mathcal{S}-\mathcal{S}^*,\mathcal{T}-\mathcal{T}^*).$$

Therefore, at least one of the crossing densities $\delta(\mathcal{S},\mathcal{T}-\mathcal{T}^*)$, $\delta(\mathcal{S}-\mathcal{S}^*,\mathcal{T})$, $\delta(\mathcal{S}-\mathcal{S}^*,\mathcal{T}-\mathcal{T}^*)$ exceeds

$$c_1 := c\frac{330^2}{330^2-1}.$$

In other words, there exist two subsystems, $\mathcal{S}_1 \subset \mathcal{S}$, $\mathcal{T}_1 \subset \mathcal{T}$, with $|\mathcal{S}_1| \geq m/330$, $|\mathcal{T}_1| \geq n/330$ such that $\delta(\mathcal{S}_1,\mathcal{T}_1) \geq c_1$.

Applying Lemma 3.2 to $\mathcal{S}_1$ and $\mathcal{T}_1$, we obtain two subsystems $\mathcal{S}^{**} \subset \mathcal{S}_1$, $\mathcal{T}^{**} \subset \mathcal{T}_1$, such that $|\mathcal{S}^{**}| \geq m/330^2, |\mathcal{T}^{**}| \geq n/330^2$, and $\delta(\mathcal{S}^{**},\mathcal{T}^{**})$ is either 1 or 0. Again, we can assume that $\delta(\mathcal{S}^{**},\mathcal{T}^{**}) = 0$, otherwise we are done. As before, we can find two subsystems, $\mathcal{S}_2 \subset \mathcal{S}_1$, $\mathcal{T}_2 \subset \mathcal{T}_1$, with $|\mathcal{S}_2| \geq m/330^2$, $|\mathcal{T}_2| \geq n/330^2$ such that

$$\delta(\mathcal{S}_2, \mathcal{T}_2) \geq c_2 := c\left(\frac{330^2}{330^2-1}\right)^2.$$

Since the crossing density between any two sets is at most 1, after some

$$k \leq \frac{\log \frac{1}{c}}{\log \frac{330^2}{330^2-1}}$$

steps, this procedure will terminate. That is, when we apply Lemma 3.2 for the k-th time, we obtain two subsystems $\mathcal{S}' \subseteq \mathcal{S}$, $\mathcal{T}' \subseteq \mathcal{T}$ such that $|\mathcal{S}'| \geq m/330^k$, $|\mathcal{T}'| \geq n/330^k$, and $\delta(\mathcal{S}', \mathcal{T}') = 1$. Thus, every element of $\mathcal{S}'$ crosses all elements of $\mathcal{T}'$, and $|\mathcal{S}'| \geq c^A m$, $|\mathcal{T}'| \geq c^A n$, where

$$A \leq \frac{\log 330}{\log \frac{330^2}{330^2-1}} < 10^6.$$

This completes the proof of Theorem 3.3 in the case when m and n are powers of 330. Otherwise, using an easy averaging argument, we can find $\mathcal{S}_0 \subseteq \mathcal{S}$, $\mathcal{T}_0 \subseteq \mathcal{T}$, whose sizes are powers of 330, $|\mathcal{S}_0| \geq m/330$, $|\mathcal{T}_0| \geq n/330$, and $\delta(\mathcal{S}_0, \mathcal{T}_0) \geq c$. Applying the above argument to $\mathcal{S}_0$ and $\mathcal{T}_0$, the result follows. $\square$

Proof of Theorem 2: Assume, for simplicity, that n is even. Given a system of n segments in general position in the plane, which determine at least cn^2 crossings, one can partition it into two equal parts so that the crossing density between them is at least $2c$ (because every graph has a bipartite subgraph containing at least half of its edges). Applying Theorem 3.3 to these parts, the result follows. $\square$

Theorem 3 can be established analogously, by repeated application of Lemma 3.2. However, here we deduce it from Theorems 2 and 3.3.

Proof of Theorem 3: Let $\mathcal{S}$ be a set of n segments in general position in the plane with at least cn^2 non-crossing pairs. For any $s \in \mathcal{S}$, let $\ell(s)$ denote the supporting line of s. The set $\ell(s) \setminus s$ consists of two half-lines; denote them by $h_1(s)$ and $h_2(s)$. Let $\mathcal{H}_1 := \{h_1(s) : s \in \mathcal{S}\}$, $\mathcal{H}_2 := \{h_2(s) : s \in \mathcal{S}\}$,

$\mathcal{T} := \mathcal{S} \cup \mathcal{H}_1 \cup \mathcal{H}_2$. Further, for any $h \in \mathcal{H}_1 \cup \mathcal{H}_2$, let $s(h)$ be the unique segment $s \in \mathcal{S}$, for which $h_1(s)$ or $h_2(s)$ is equal to s.

Note that if two segments $s, t \in \mathcal{S}$ do not cross each other, then the crossing between their supporting lines, $\ell(s)$ and $\ell(t)$, gives rise to a crossing between a pair of elements of $\mathcal{T}$, involving at least one half-line. Therefore, the number of crossing pairs in $\mathcal{T}$ involving at least one half-line is at least cn^2. There are three possibilities:

1. for some $i = 1, 2$, the number of crossing pairs in $\mathcal{H}_i$ is at least $cn^2/5$;
2. the number of crossing pairs between $\mathcal{H}_1$ and $\mathcal{H}_2$ is at least $cn^2/5$;
3. for some $i = 1, 2$, the number of crossing pairs between $\mathcal{H}_i$ and $\mathcal{S}$ is at least $cn^2/5$.

In Case 1, applying Theorem 2 to $\mathcal{H}_i$, we obtain two subsystems, $\mathcal{H}_{i1}, \mathcal{H}_{i2} \subset \mathcal{H}$, whose sizes are at least $\frac{(2c/5)^A}{660} n > \frac{(c/5)^A}{330}$, and every half-line in $\mathcal{H}_{i1}$ crosses all half-lines in $\mathcal{H}_{i2}$. Then $\mathcal{S}_1 := \{s(h) : h \in \mathcal{H}_{i1}\}$ and $\mathcal{S}_2 := \{s(h) : h \in \mathcal{H}_{i2}\}$ meet the requirements in Theorem 3.

In Case 2, apply Theorem 3.3 to obtain $\mathcal{H}'_1 \subseteq \mathcal{H}_1$, $\mathcal{H}'_2 \subseteq \mathcal{H}_2$, whose sizes are at least $\frac{(c/5)^A}{330} n$, and every element of $\mathcal{H}'_1$ crosses all elements of $\mathcal{H}'_2$. Setting $\mathcal{S}_1 := \{s(h) : h \in \mathcal{H}'_1\}$, and $\mathcal{S}_2 := \{s(h) : h \in \mathcal{H}'_2\}$, the result follows. Case 3 can be treated similarly. $\Box$

4 Concluding remarks

First we show how Theorems 4 and 5 follow from the previous results.

Proof of Theorem 4: Let G be a geometric graph with n vertices and at least cn^2 edges. The next result of Ajtai, Chvátal, Newborn, Szemerédi [ACNS82] and, independently, Leighton [L83] (see also [PA95], [PT97]) implies that there are at least $\frac{c}{64} e^2$ crossings pairs of edges.

Lemma A. *Let G be a geometric graph with n vertices and $e > 4n$ edges, for some $c > 0$. Then G has at least $\frac{e^3}{64n^2}$ crossing pairs of edges.*

Thus, we can apply Theorem 2 to the system $\mathcal{S} = E(G)$. We obtain two subsets $E_1, E_2 \in E(G)$ such that every edge in E_1 crosses all edges in E_2, and $|E_1| = |E_2| \geq \frac{(c/32)^A}{336} cn^2 > (c/32)^{A+2} \binom{n}{2}$. $\Box$

Theorem 5 can be proved similarly. The only difference is that, instead of Theorem 2 and Lemma A, we have to use Theorem 3 and

Lemma B. [P91] *Let G be a geometric graph with n vertices and $e \geq 3n/2$ edges, for some $c > 0$. Then G has at least $\frac{4e^3}{27n^2}$ pairs of edges that do not cross and do not share an endpoint.*

The above theorems can also be established using Szemerédi's Regularity Lemma [S78]. However, then the dependence on c of the sizes of the homogeneous subsystems whose existence is guaranteed by our results gets much worse.

According to an old theorem of Kővári, Sós, and Turán [KST54], every graph with n vertices and at least cn^2 edges has a complete bipartite subgraph with $c' \log n$ vertices in its classes, where $c' > 0$ is a suitable constant depending on c. This immediately implies that Theorem 2 holds with the much weaker bound $c' \log n$ instead of $c'n$.

For some computational aspects of recognizing intersection graphs of segments, see [KM94].

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