Planar crossing numbers of graphs embeddable in another surface

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Abstract

Let G be a graph of n vertices with maximum degree d that can be drawn without crossing in a closed surface of Euler characteristic χ. It is proved that then G can be drawn in the plane with at most $c_\chi dn$ crossings, where c_χ is a constant depending only on χ. This result, which is tight up to a constant factor, is strengthened and generalized to the case when there is no restriction on the degrees of the vertices.

1 Introduction and preliminaries

Let G be a graph with no loops or parallel edges. A *drawing* or *embedding* of G on a closed surface S is a representation of G such that the vertices are represented by points and the edges are represented by (possibly crossing) non-selfintersecting continuous arcs in S, connecting the corresponding point pairs and not passing through any other vertex. The *crossing number* (or planar crossing number) of G, $\mathrm{cr}(G)$, is defined as the minimum number of edge crossings over all drawings of G in the plane (or equivalently, on the sphere).

Suppose that G has n vertices and e edges, and suppose that it permits a crossing-free drawing on a closed surface S. How large can $\mathrm{cr}(G)$ be? Obviously, we have $\mathrm{cr}(G) < \binom{e}{2}$, and this order of magnitude can be attained if S is not the sphere (plane). To see this, take five vertices and connect any pair of them by $\frac{e}{20}$ vertex-disjoint paths of lengths two. Clearly, this graph can be embedded in any closed surface other than the sphere, but no matter how we draw it in the plane, every subdivision of K_5 gives rise to a crossing. Therefore, the number of crossings must be at least $\frac{e^2}{400}$.

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It was conjectured by Peter Brass that this estimate can be substantially improved if we impose an upper bound on the degree of the vertices. This conjecture has been verified in [PT05], for any *orientable* surface S. The aim of this note is to prove Brass' conjecture in full generality.

Theorem 1. *Let G be a graph of n vertices with maximum degree d, and suppose that G has a crossing-free drawing on the closed surface S of Euler characteristic χ. Then the planar crossing number of G satisfies*

$$\mathrm{cr}(G) \leq 10^{20-7\chi} dn.$$

It is not hard to see that for any fixed χ, this bound is best possible up to a constant factor. For instance, if S is the simplest *nonorientable* surface, the projective plane, consider the *grid* graph H_k defined on the vertex set $V(H_k) = \{(i,j) : 1 \leq i, j \leq k\}$, in which (i,j) and (i',j') are connected by an edge if and only if $|i-i'| + |j-j'| = 1$. Let H'_k denote the graph obtained from H_k by connecting each vertex $(1,j)$ to $(k, k-j+1)$, and each vertex $(i,1)$ to $(k-i+1, k)$, for all $1 \leq j \leq k$ and $1 \leq i \leq k$. Furthermore, let G be the graph obtained from H'_k by replacing every edge with $\lceil d/4 \rceil$ paths of length two connecting the same pair of vertices. Clearly, G is a graph of $n = k^2(2\lceil d/4 \rceil + 1)$ vertices of maximum degree at most d, and it can be embedded in the projective plane without any crossing. It is not hard to verify, and follows from a recent result of Gitler, Leanos, and Salazar [GLS06] that $\mathrm{cr}(G) \geq cdn$ holds, for a suitable constant $c > 0$. Instead of Theorem 1, we prove a stronger result.

Theorem 2. *Let G be a graph of n vertices with degrees $d_1, d_2, \ldots, d_n$, and suppose that G has a crossing-free drawing on a closed surface S of Euler characteristic χ. Then we have*

$$\mathrm{cr}(G) \leq 10^{19-7\chi} \sum_{i=1}^{n} d_i^2.$$

In Section 2, first we reduce Theorem 1 to Theorem 2. Then we reduce Theorem 2 to another statement (Theorem 3 stated there), which, in turn, will be established in the last two sections.

For the proof, we need the well known classification of *closed surfaces* (that is, compact surfaces with no boundary), discovered in the 1860's [H02], [MT01]. According to this, every closed surface is homeomorphic to one of the surfaces S_i or N_i $(i = 0, 1, 2, \ldots)$. Here $S_0 = N_0$ is the sphere (plane), and for every $i > 0$, S_i and N_i can be obtained from S_{i-1} and N_{i-1} by attaching a *handle* and a *crosscap*, respectively. The surface S_i is *orientable*, and its *Euler characteristic* is $\chi(S_i) = 2 - 2i$ $(i \geq 0)$. The surface N_i is *nonorientable*, and we have $\chi(N_i) = 2 - i$ $(i > 0)$. Hence, the Euler characteristic of any surface is an integer that does not exceed *two*.

Fix a closed simple curve γ in S, and consider a small neighborhood of γ. It is topologically equivalent either to a *strip* or to a *Möbius strip*. In the first case, we say that γ is *orientation preserving*, in the second case, it is *orientation reversing*. The surface S is orientable if and only if it contains no orientation reversing curve. If $S \setminus \gamma$ is connected, then γ is called a *nonseparating* curve. Otherwise, $S \setminus \gamma$ consists of two connected components and γ is said to be *separating.*

For the purpose of this paper, we classify the closed curves as follows.

Type 1: γ *is separating.*

In this case, γ is necessarily *orientation preserving*, and any closed curve intersects it an *even* number of times. Cutting S along γ, we obtain two surfaces, both bounded by a closed curve. Attach a disk to each of these boundaries. If γ was contractible to a point, then we obtain a sphere and a surface homeomorphic to S. Otherwise, we obtain two closed surfaces whose Euler characteristics are larger than $\chi(S)$.

Type 2: γ *is nonseparating and orientation preserving.*

In this case, there exists another closed curve which intersects γ exactly once. Cutting S along γ, and attaching a disk to each of the resulting boundary curves, we obtain a compact surface S' with Euler characteristic $\chi(S) + 2$.

Type 3: γ *is orientation reversing, and hence nonseparating.*

This case can occur only for nonorientable surfaces. There exists another closed curve which intersects γ exactly once. Cutting S along γ, we obtain a surface with one closed boundary curve. Attach a disk to it. The Euler characteristic of the resulting surface S' is $\chi(S) + 1$. Note that S' may be orientable or nonorientable.

For any graph G, let $n(G)$, $e(G)$, and $\sigma(G)$ denote the number of vertices of G, edges of G, and the sum of the squares of their degrees, respectively.

A crossing-free drawing of G on a closed surface S is called a *cellular embedding* (or a *two-cell embedding*) if every connected component of $S \setminus G$ is homeomorphic to an open disk. It follows from the Euler-Poincare formula that, if G has a cellular embedding in S, then we have $e(G) \leq 3n(G) - 3\chi(S)$.

2 Reducing Theorems 1 and 2 to Theorem 3

For both Theorems 1 and 2 we can assume that G is connected, otherwise we can argue separately for its components. We can also assume that $\chi < 2$, otherwise there is nothing to prove. Therefore, we have $10^{20-7\chi} dn > 1000n$ and similarly $10^{19-7\chi} \sum_{i=1}^{n} d_i^2 > 1000n$. On the other hand, clearly, $\mathrm{cr}(G) < 1000n$ holds for every graph G of at most 10 vertices. Hence we can assume that $n > 10$. Suppose that $|\chi| \geq n/10$. Then we have $10^{20-7\chi} dn > 10^{7n/10} > n^4 > \mathrm{cr}(G)$, and similarly $10^{19-7\chi} \sum_{i=1}^{n} d_i^2 > 10^{7n/10} > n^4 > \mathrm{cr}(G)$, so there is nothing to prove. So, we can and will assume in the sequel that $|\chi| \leq n/10$.

First we show that Theorem 2 implies Theorem 1. Consider a crossing-free drawing of G on a closed surface S of *maximum* Euler characteristic χ. Notice that this must be a *cellular* embedding. Indeed, if a connected component of $S \setminus G$ is not homeomorphic to a disk, then replacing such a cell by a disk, we obtain a surface S' whose Euler characteristic is larger than χ.

Applying the consequence of the Euler-Poincare formula mentioned at the end of Section 1, we obtain that $e(G) \leq 3n - 3\chi < 4n$.

Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degrees of the vertices. By Theorem 2, we have

$$\mathrm{cr}(G) \leq 10^{19-7\chi} \sum_{i=1}^{n} d_i^2 \leq 10^{19-7\chi} d_1 \sum_{i=1}^{n} d_i \leq 10^{19-7\chi} d_1 \cdot 2e(G) \leq 10^{19-7\chi} d_1 \cdot 10n = 10^{20-7\chi} d_1 n,$$

and Theorem 1 follows $\Box$.

Next we turn to the proof of Theorem 2. Recall that $\sigma(G)$ denotes the sum of degrees of the vertices of G. In this case, it is sufficient to prove the statement for *two-connected* graphs. Indeed, if G is disconnected or has a cut vertex, then it can be obtained as the union of two graphs G_1 and G_2 with n_1 and n_2 vertices that have at most one vertex in common. Arguing for G_1 and G_2 separately, we obtain by induction that

$$\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2) \leq 10^{19-7\chi}(\sigma(G_1) + \sigma(G_2)) \leq 10^{19-7\chi}\sigma(G),$$

as required.

Consider a crossing-free drawing of G on a closed surface S of *maximum* Euler characteristic χ. As in the proof of Theorem 1, it follows from the maximality that this is a *cellular* embedding.

If our drawing is not a triangulation, then by adding $O(n)$ extra vertices and edges, we can turn it into one so that the maximum degree of the vertices increases by at most a factor of three. We have to apply the following observation that can be established by a slight modification of the proof of Lemma 2.1 in [PT05], where the same statement is proved under the assumption that S is orientable.

Lemma 2.1. *Let G be a two-connected graph with n vertices, each of degree at most d ($d \geq 3$). Suppose that G has a cellular embedding into a closed surface with Euler characteristic χ. Any such embedding can be extended to a triangulation of S with at most $19n - 18\chi$ vertices of maximum degree at most $3d$.* $\Box$

For any graph G and for any $\chi \leq 2$, let $\mathrm{cr}_\chi(G)$ be the minimum number of edge crossings over all drawings of G on a surface of Euler characteristic χ. Using this notation, we have that $\mathrm{cr}_2(G) = \mathrm{cr}(G)$.

Theorem 2 provides an upper bound on the crossing number of graphs that can be embedded into a surface of Euler characteristic χ. We show that this bound can be deduced by repeated application of the following result. In each step, we increase the Euler characteristic of the surface by at least one.

Theorem 3. *Let G be a two-connected graph with $\mathrm{cr}_\chi(G) = 0$ for some $\chi < 2$. Then we have $\mathrm{cr}_{\chi+1}(G) \le 10^5\sigma(G)$.*

Proof of Theorem 2 using Theorem 3. The proof is by reverse induction on $\chi(S)$. Theorem 2 is trivial for $\chi(S) = 2$, that is, if S is the sphere. Let $\chi < 2$, and suppose that we have already proved Theorem 2 for all $\chi' > \chi$. Consider a crossing-free drawing of G on a closed surface S with $\chi(S) = \chi$. According to Theorem 3, G can be drawn with at most $10^5\sigma(G)$ crossings on another closed surface S' of Euler characteristic $\chi(S') \ge \chi + 1$. Place a new vertex at each crossing and denote the resulting graph by G'. Since every new vertex has degree *four*, we have

$$\sigma(G') \le \sigma(G) + 4^2\mathrm{cr}_{\chi+1}(G) \le \sigma(G) + 4^2 \cdot 10^5\sigma(G) = 1600001\sigma(G).$$

Applying the induction hypothesis to G', we obtain that there exists a drawing of G' in the plane with

$$\mathrm{cr}(G') \le 10^{19-7\chi-7}\sigma(G') \le 10^{12-7\chi} \cdot 1600001\sigma(G)$$

crossings. Consider such a drawing, and replace all new vertices of G' by a crossing between the corresponding edges of G. Then the number of crossings in the resulting planar drawing of G is at most

$$10^{12-7\chi} \cdot 1600001\sigma(G) + 10^5\sigma(G) < 10^{19-7\chi}\sigma(G),$$

as required. □

It remains to prove Theorem 3. This will be done in the last two sections.

3 Proof of Theorem 3: Cutting along a cycle of type 2

As in Section 1, after Theorem 1, define a *grid* of size $k \times k$ as the cross product $P_k \times P_k$ of two paths of length k. The *boundary* of the grid consists of all vertices of $P_k \times P_k$ whose degree is less than *four*.

Since $\mathrm{cr}_{\chi'}(G) \le \mathrm{cr}_\chi(G)$ for every $\chi' > \chi$, to establish Theorem 3, it is sufficient to consider a crossing-free drawing of G in a surface S of *maximum* Euler characteristic χ. As we have seen at the beginning of Section 2, this implies that we have a *cellular embedding* of G, that is, every cell of $S \setminus G$ is homeomorphic to an open disk. Moreover, since every graph G has a crossing-free drawing on S_g with $g = e(G) - 1$ [MT01], we can assume that $\chi > -2e(G) \ge -\sigma(G)/2$.

In the above cellular embedding of G in S, replace each vertex v with degree $d(v) > 4$ by a grid of size $d(v) \times d(v)$, lying in a small neighborhood of v, and connect the edges incident to v to distinct vertices on the boundary of the grid, without changing their cyclic order. Denote the resulting graph and its embedding in S by G'. Clearly, G' has at most $\sigma(G)$ vertices, each of degree at most *four*. This is still a cellular embedding, so that we can apply Lemma 2.1 to turn G' into a triangulation G'' with at most

$$19\sigma(G) - 18\chi < 28\sigma(G)$$

vertices, each of degree at most *twelve*.

Restricting G' and G'' to any grid substituting for a vertex in G, the only difference between them is that each quadrilateral face in G' is subdivided by one of its diagonals into two triangles in G''. Color all edges along the boundaries of the grids *blue*, and all other grid and diagonal edges of G'' that lie in the interior of some grid *red*. Color all other edges *green*. Surface S is not the sphere, so it contains at least one closed curve of type 2 or 3, that is, at least one nonseparating closed curve (see the end of Section 1). Since G'' determines a triangulation of S, any closed curve is homotopic to a cycle of G'' (see [MT01]). In particular, G'' has a nonseparating cycle.

In the rest of this paper, let C denote a fixed cycle of G'' of type 2 or 3, with the *smallest* number k of vertices. In this section, we settle the case when C *is of type 2*, that is, C is nonseparating and orientation preserving. The other case, when C is of type 3, will be discussed in Section 4. The two cases of the proof of Theorem 3 are similar. However, they differ at some key points, so in spite of the necessary repetition, they are presented separately.

Lemma 3.1. *Let C be of type 2. Then its length k satisfies $k \le 11\sqrt{\sigma(G)}$.*

Proof of Lemma 3.1. Fix an orientation of C and let $v_1, \dots, v_k$ be the vertices of C, in cyclic order. Let E^l (and E^r) denote the set of edges not belonging to C that are incident to at least one vertex of C and in a small neighborhood of this vertex lie on its left-hand side (respectively right-hand side). Replace C by two copies, C^r and C^l, lying on its right-hand side and left-hand side, and denote the vertices of C^r and C^l by $v_1^r, \dots, v_k^r$ and $v_1^l, \dots, v_k^l$, respectively. Connect each edge of E^r and E^l to the corresponding vertex of C_r and C_l. Cut S along C, and attach two disks to the surface: one along C_r and one along C_l. The resulting surface S' satisfies $\chi(S') = \chi(S) + 2$, and it contains a crossing-free drawing G_1 of a graph slightly different from G''. Contract $v_1^r, \dots, v_k^r$ resp. $v_1^l, \dots, v_k^l$ into a new vertex V^r resp. V^l. If during this procedure multiple edges are create, remove all repetitions, that is, keep only one edge between any pair of points. Denote the resulting graph by G_2. It has at most $28\sigma(G)$ vertices, and it is drawn on S' as a triangulation.

Claim 3.2. There is no path of length smaller than $k/2$ in G_2, connecting V^r and V^l.

Proof. Suppose for contradiction that there is such a path P. It corresponds to a path P of the same length connecting some v_i^r and v_j^l in G_2. Clearly, i and j are at distance at most $k/2$ in the cyclic order. Assume without loss of generality that $i \le j \le i + k/2$. Extend the path P by adding $v_j^l v_{j-1}^l \dots v_i^l$. The new path, denoted by $\tilde{P}$, connects v_i^r to v_i^l and it has length $\tilde{l} < k$. Clearly, $\tilde{P}$ corresponds to a cycle in G'' with the same length $\tilde{l} < k$. Since this cycle intersects C^r exactly once, very close to v_i, it is nonseparating, therefore, it is either of type 2 or of type 3. This is a contradiction, because, according to our assumption, G'' has no nonseparating cycle shorter than C. $\square$

Return to the proof of Lemma 3.1. By Menger's theorem, the maximum number p of (internally) vertex-disjoint paths connecting V^r and V^l in G_2 is equal to the minimum number of vertices whose deletion separates V^r from V^l. Choose p such separating vertices $u_1, \dots, u_p$, and let K denote the set of all edges incident to $u_1, \dots, u_p$. With a slight abuse of notation, we use the same letter K to denote the union of these edges, as a point set, including the endpoints of the edges.

Claim 3.3. The set K contains a nonseparating cycle in S, whose length is at most $2p$.

Proof. Suppose that there is a curve δ connecting V^r and V^l in S', which does not meet K. Orient δ towards V^l. Let $e_1, \dots, e_q$ denote the sequence of edges of G, in the order they are intersected by δ, and let $w_1, \dots, w_q$ denote the right endpoints of these edges, with respect to the orientation of δ. Using the fact that $\tilde{G}$ is a triangulation, we obtain that for every i, $1 \le i < q$, the points w_i and w_{i+1} are either identical or adjacent. Therefore, $V^r w_1 w_2 \dots w_q V^l$ is a path in G_2 connecting V^r to V^l. Using the fact that δ does not intersect K, this path must avoid all edges incident to the separating vertices $u_1, \dots, u_p$. Thus, $V^r w_1, \dots, w_q V^l$ avoids the points $u_1, \dots, u_p$, a contradiction.

Thus, we can conclude that K contains a cycle that separates V^r and V^l on the surface S'. Since every edge that belongs to K is incident to at least one of the points $u_1, \dots, u_p$, the length of shortest such cycle $\tilde{C}$ is at most $2p$. We claim that $\tilde{C}$ is a *nonseparating* cycle in S. By the minimality of $\tilde{C}$, there is a curve γ connecting V^r and V^l on S', which intersects $\tilde{C}$ exactly once. As in the proof of Claim 3.2, we can use γ and C to construct a closed curve on S, which intersects $\tilde{C}$ exactly once. This implies that $\tilde{C}$ is a nonseparating cycle in S. $\Box$

Now we are in a position to complete the proof of Lemma 3.1. Since the length of the shortest nonseparating cycle (in $G'' \subset S$) is k, it follows from Claim 3.3 that there are $p \ge k/2$ vertex-disjoint paths connecting V^r and V^l. By Claim 3.2, the length of each of them is at least $k/2$. Using the fact that G'' has at most $28\sigma(G)$ vertices, we can conclude that $k^2/4 \le 28\sigma(G)$, so that $k \le 11\sqrt{\sigma(G)}$, as required. The proof of Lemma 3.1 is complete. $\Box$

Without increasing the length of C too much, we can replace all of its red edges by blue ones. Indeed, the first vertex and the last vertex of any maximal red path in C must belong to the boundary of the same grid. Replace each such path by the shortest blue path connecting its first and last vertices along the boundary of the grid containing them. The resulting blue cycle C' is also of type 2, and its length m is at most $2k$. We can assume without loss of generality that C' does not intersect itself. Fix an orientation of C'.

Proceed now with C' as we did with C in the proof of Lemma 3.1. To simplify the notation, let us again denote by E_l (by E_r) the set of edges not belonging to C', which are incident to at least one vertex of C' and in a small neighborhood of this vertex lie on the left-hand side (right-hand side) of C'. Replace again C' by two of its copies, C_r and C_l, lying on its right-hand side and left-hand side, and denote the vertices of C^r and C^l by $v_1^r, \dots, v_m^r$ and $v_1^l, \dots, v_m^l$, respectively. Connect every edge of E_r (and E_l) to the corresponding vertex of C_r (and C_l). Cut S along C, and attach to the surface a disk along each side of the cut. For the resulting surface S', we have $\chi(S') = \chi(S) + 2$, and S' contains a *crossing-free drawing* G_1 of a graph slightly different from

G''. (Note that G_1 may be different from the drawing we denoted by G_1 in the proof of Lemma 3.1.) To obtain a drawing of G'' from G_1, we have to remove C_l and (re)connect the edges of E_l to the corresponding vertices of C_r, without creating too many crossings.

As before, contract $v_1^r, \dots, v_m^r$ and $v_1^l, \dots, v_m^l$ into two new vertices, V^r and V^l, and remove the possibly arising repeated edges. The resulting graph G_2 has at most $28\sigma(G)$ vertices, and it is drawn on S' as a triangulation. By Menger's theorem, the maximum number p of vertex-disjoint paths connecting V^r and V^l in G_2 is equal to the minimum number of vertices whose deletion separates V^r from V^l, and again we have $p \geq k/2$. Therefore, the length of the shortest connecting path is at most $28\sigma(G)/p \leq 56\sigma(G)/k$. Such a shortest path P corresponds to a path of the same length connecting some vertex v_i^r to a vertex v_j^l in G_1. As before, we can replace all red edges of P by blue edges, without increasing its length by more than a factor of *two*. Let P' denote the resulting blue path, whose length ℓ is therefore smaller than $112\sigma(G)/k$.

Delete the vertices of C_l. We want to use P' as a "corridor" through which we can "pull" all edges in E_l, and connect them to the corresponding vertices of C_r. Observe that for every vertex v of P', all *red* edges incident to v are on the same side of P'. Therefore, when we extend the edges of E_l along P', we can avoid crossing any red edge. To achieve this, the extended edges of E_l may have to cross P' several times.

Let us count the number of crossings in the resulting drawing. Each vertex of C' has degree at most 12, so it is adjacent to at most 10 edges of E_l. Since $|C'| \leq 2k$, we have $|E_l| \leq 20k$. Pulling the edges in E_l along P', we can create at most 12 crossings per edge, in the neighborhood of each vertex of P'. The total number of these crossings is at most $12l \cdot 20k \leq 26880k \cdot \sigma(G)/k = 26880\sigma(G)$. In addition, we may have to create some crossings between the edges of E_l. The number of such crossings cannot exceed $\binom{|E_l|}{2}$. Using Lemma 3.1, we obtain that $\binom{\ell}{2} \leq \binom{20k}{2} < 24200\sigma(G)$. Summarizing, during the whole procedure we created at most $X = 51080\sigma(G)$ crossings.

Deleting the extra vertices and edges from G_1 and collapsing each grid into a vertex, we obtain a drawing of G on S', in which the number of crossings cannot exceed X. This concludes the proof of Theorem 3 in the case when the shortest cycle $C \subset G''$ homotopic to γ was of type 2.

4 Proof of Theorem 3: Cutting along a cycle of type 3

It remains to prove Theorem 3 in the case when the shortest cycle $C \subset G''$ homotopic to γ is of type 3, that is, orientation reversing, and hence nonseparating. (See the beginning of Section 3, above Lemma 3.1.)

Lemma 4.1. *Let C be of type 3. Then its length k satisfies $k \leq 35\sqrt{\sigma(G)}$.*

Proof of Lemma 4.1. Let $v_1, \dots, v_k$ denote the vertices of C, in cyclic order. Suppose for simplicity that k is divisible by *four*. Contract $v_1, \dots v_{k/4-1}$ into a new vertex V_1, contract

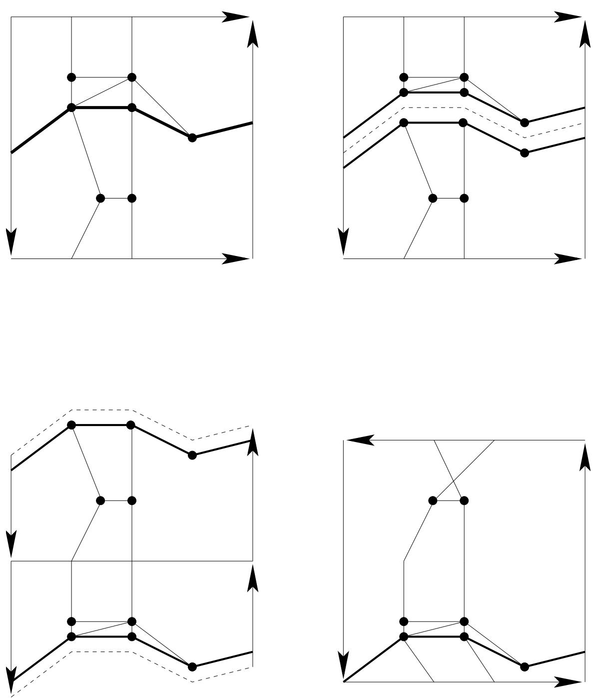

Figure 1: Cut the Klein bottle along a cycle of type 3 to get a drawing in the projective plane.

$v_{k/4+1}, \ldots v_{k/2-1}$ into a vertex V_2, contract $v_{k/2+1}, \ldots v_{3k/4-1}$ into V_3, and $v_{3k/4+1}, \ldots v_{k-1}$ into V_4. Remove the possibly arising multiplicities by keeping only one edge between any pair of vertices. The resulting graph $\tilde{G}$ has at most $28\sigma(G)$ vertices, and it is drawn on S as a triangulation. Let $\tilde{C}$ denote the cycle $V_1 v_{k/4} V_2 v_{k/2} V_3 v_{3k/4} V_4 v_k \subset \tilde{G}$ that corresponds to $C \subset G$. Clearly, $\tilde{C}$ is also orientation reversing.

Claim 4.2. (i) There is no path of length smaller than $k/4$ in $\tilde{G} \setminus \{V_2, V_4\}$, connecting V_1 and V_3.

(ii) There is no path of length smaller than $k/4$ in $\tilde{G} \setminus \{V_1, V_3\}$, connecting V_2 to V_4.

Proof. (i) Suppose for contradiction that there is such a path $\tilde{P}$. It corresponds to a path P of the same length connecting v_i and v_j in G, for some $1 \leq i \leq k/4 - 1$, $k/2 + 1 \leq j \leq 3k/4 - 1$. The vertices v_i and v_j divide C into two intervals. One of these intervals together with P forms an orientation reversing cycle, which is shorter than C, a contradiction.

Part (ii) of the claim can be proved analogously. $\square$

Return to the proof of Lemma 4.1. By Menger's theorem, the maximum number p of vertex-disjoint paths connecting V_1 and V_3 in $\tilde{G}$ is equal to the minimum number of vertices whose deletion separates V_1 from V_3. Choose p such separating vertices, $u_1, \ldots, u_p$. As in the proof of Lemma 3.1, let K stand for the collection of all edges incident to $u_1, \ldots, u_p$, as well as the closed point set obtained by taking the union of these edges.

Claim 4.3. There is a path of length at most $2p$ in K, connecting V_2 and V_4.

Proof. Suppose that there is a curve δ connecting V_1 and V_3, which is disjoint from K. Orient δ towards V_3. Let $e_1, \ldots, e_q$ denote the sequence of edges of G intersected by δ, and let $w_1, \ldots, w_q$ denote the right endpoints of these edges, with respect to the orientation of δ. Since $\tilde{G}$ is a triangulation, it follows that for every i, $1 \leq i < q$, the vertices w_i and w_{i+1} are either identical or adjacent. Therefore, $V_1 w_1 w_2 \ldots w_q V_3$ is a path in $\tilde{G}$ connecting V_1 and V_3. By definition, δ does not intersect K, hence the path $V_1 w_1, \ldots, w_q V_3$ avoids all points $u_1, \ldots, u_p$, which is impossible. Thus, K separates V_1 and V_3 on the surface S. The separating set $\{u_1, \ldots, u_p\}$ must block $V_1 v_{k/4} V_2 v_{k/2} V_3$ and $V_1 v_k V_4 v_{3k/4}$, the pair of paths connecting V_1 and V_3 along $\tilde{C}$. It follows that K contains both V_2 and V_4, and hence a whole path connecting V_2 and V_4. Every edge that belongs to K incident to at least one u_i $(1 \leq i \leq p)$. This yields that the length of any simple path in K between V_2 and V_4 is at most $2p$. $\Box$

To complete the proof of Lemma 4.1, recall that p is the maximum number of vertex-disjoint paths connecting V_1 and V_3 in $\tilde{G}$. One of them passes through V_2 and one of them through V_4. By Claim 4.2(i), each of the other $p-2$ paths has length at least $k/4$. According to Claim 4.3, there is a path connecting V_2 and V_4 in $\tilde{G} \setminus \{V_1, V_3\}$, whose length is at most $2p$. By Claim 4.2(ii), its length is at least $k/4$, so we have $p \geq k/8$. Therefore, $28\sigma(G) \geq (k/8 - 2) \cdot k/4$, so $\sqrt{28 \cdot 32\sigma(G)} \geq k - 16$. Since G has at least 10 vertices and it is two-connected, we have that $\sigma(G) \geq 40$, and then it follows by a routine calculation that $k < 35\sqrt{\sigma(G)}$. This concludes the proof of Lemma 4.1. $\Box$

The end of the proof is very similar to the end of the proof (described in Sectiom 3) in the case when C is of type 2. Without increasing the length of C by more than a factor of *two*, we can replace all red edges of C by blue edges. Denote by C' the resulting blue cycle of length at most $2k$, and fix an orientation of C'. Let E_l (and E_r) denote the set of edges not belonging to C' that are incident to at least one vertex of C' and in a small neighborhood of this vertex lie on the left-hand side (right-hand side) of C'. Replace each vertex v_i of C by two copies, v_i^l and v_i^r, on the left-hand and right-hand sides of C', respectively. Connect each edge of E_r and E_l) to the corresponding vertex v_i^r or v_i^l.

Cut S along C. Since C is an *orientation reversing* cycle, we obtain one boundary curve. Attach a disk to the surface along this boundary curve. The resulting surface S' has Euler characteristic $\chi(S') = \chi + 1$, and it contains a crossing-free drawing G_1 of a graph slightly different from G''. To obtain a drawing of G'' from G_1, we have to remove $v_1^l, \ldots, v_k^l$ and (re)connect the edges of E_l to the corresponding vertices among $v_1^r, \ldots, v_k^r$, without creating too many crossings. Since each vertex has degree at most *twelve*, we have to reconnect at most $12k$ edges. Clearly, this can be done within the disk attached to the surface along the cut, creating at most one crossing between any pair of edges and no crossing with the other edges of G''. Therefore, the total number of crossings is at most

$$X \leq \binom{12k}{2} < 72k^2 < 72 \cdot 35^2 \sigma(G) < 90000\sigma(G).$$

Deleting the extra vertices and edges from G'' and collapsing each grid into a vertex, we obtain a drawing of G on S', in which the number of crossings cannot exceed X. This concludes the proof of Theorem 3 in the case when C is of type 3.

Remark. The coefficients $10^{20-7\chi}$ and $10^{19-7\chi}$ in Theorems 1 and 2, respectively, are very far from being optimal. With a slightly more careful calculation they can be reduced to significantly smaller exponential functions, but it is very likely that the optimal coefficients are polynomials, or even linear functions of χ.

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