Applications of the Crossing Number

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Abstract

Let G be a graph of n vertices that can be drawn in the plane by straight-line segments so that no $k+1$ of them are pairwise crossing. We show that G has at most $c_k n \log^{2k-2} n$ edges. This gives a partial answer to a dual version of a well-known problem of Avital–Hanani, Erdős, Kupitz, Perles, and others. We also construct two point sets $\{p_1, \ldots, p_n\}$, $\{q_1, \ldots, q_n\}$ in the plane such that any piecewise linear one-to-one mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ with $f(p_i) = q_i$ $(1 \leq i \leq n)$ is composed of at least $\Omega(n^2)$ linear pieces. It follows from a recent result of Souvaine and Wenger that this bound is asymptotically tight. Both proofs are based on a relation between the crossing number and the bisection width of a graph.

Keywords: Crossing number, geometric graph, bisection width, triangulation.

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1 Introduction

A *geometric graph* is a graph drawn in the plane by (possibly crossing) straight-line segments i.e., it is defined as a pair of $(V(G), E(G))$, where $V(G)$ is a set of points in the plane in general position and $E(G)$ is a set of closed segments whose endpoints belong to $V(G)$.

The following question was raised by Avital and Hanani [AH], Erdős, Kupitz [K] and Perles: What is the maximum number of edges that a geometric graph of n vertices can have without containing $k+1$ pairwise disjoint edges? It was proved in [PT] that for any fixed k the answer is linear in n. (The cases when $k \leq 3$ had been settled earlier by Hopf and Pannwitz [HF], Erdős [E], Alon and Erdős [AE], O'Donnel and Perles [OP], and Goddard, Katchalski and Kleitman [GKK].)

In this paper we shall discuss the dual counterpart of the above problem. We say that two edges of G *cross* each other if they have an interior point in common. Let $e_k(n)$ denote the maximum number of edges that a geometric graph of n vertices can have without containing $k+1$ *pairwise crossing* edges. If G has no two crossing edges, then it is a planar graph. Thus, it follows from Euler's polyhedral formula that

$$e_1(n) = 3n - 6 \quad \text{for all } n \geq 3.$$

It was shown in [P] that $e_2(n) < 13n^{3/2}$ and that, for any fixed k,

$$e_k(n) = O(n^{2-1/25(k+1)^2}).$$

However, we suspect that $e_k(n) = O(n)$ holds for every fixed k as n tends to infinity. We know that the corresponding statement is true if we restrict our attention to *convex* geometric graphs, i.e., to geometric graphs whose vertices are in convex position [CP]. Our next theorem brings us fairly close to this bound for arbitrary geometric graphs.

Theorem 1.1 *Let G be a geometric graph of n vertices, containing no $k+1$ pairwise crossing edges. Then the number of edges of G satisfies*

$$|E(G)| \leq c_k n \log^{2k-2} n,$$

with a suitable constant c_k depending only on k.

The proof is based on a general result relating the crossing number of a graph to its bisection width (see Theorem 2.1). A nice feature of our approach is that we do not use the assumption that the edges of G are line segments. Theorem 1.1 remains valid for graphs whose edges are represented by arbitrary Jordan arcs such that any two arcs meet at most once (or at most a bounded number of times).

The same ideas can be used to settle the following problem. Let T_1 and T_2 be triangles in the plane, and let $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_n\}$ be two n-element point sets lying in the interior of T_1 and T_2, respectively. A *homeomorphism* f from T_1 onto T_2 is a continuous one-to-one mapping with continuous inverse. f is called *piecewise linear* if there exists a triangulation of T_1 such that f is linear on each of its triangles. The *size* of f is defined as the minimum number of triangles in such a triangulation. Recently, Souvaine and Wenger [SW] have shown that one can always find a piecewise linear homeomorphism $f : T_1 \to T_2$ with $f(p_i) = q_i$ $(1 \leq i \leq n)$ such that the size of f is $O(n^2)$. Our next result shows that this bound cannot be improved.

Theorem 1.2 *There exist a triangle T and two point sets $\{p_1, \ldots, p_n\}$, $\{q_1, \ldots, q_n\} \subseteq \mathtt{int}\ T$ such that the size of any piecewise linear homeomorphism $f : T \to T$ which maps p_i to q_i $(1 \leq i \leq n)$ is at least cn^2 (for a suitable constant $c > 0$).*

For some closely related problems consult [S] and [ASS].

2 Crossing number and bisection width

Let G be a graph of n vertices with no loops and no multiple edges. For any partition of the vertex set $V(G)$ into two disjoint parts V_1 and V_2, let $E(V_1, V_2)$ denote the set of edges with one endpoint in V_1 and the other endpoint in V_2. Define the *bisection width* of G as

$$b(G) = \min_{|V_1|, |V_2| \geq n/3} |E(V_1, V_2)|,$$

where the minimum is taken over all partitions $V(G) = V_1 \cup V_2$ with $|V_1|$, $|V_2| \geq n/3$.

Consider now a drawing of G in the plane, where the vertices are represented by distinct points and the edges are represented by Jordan arcs connecting them such that (1) no arc passes through a vertex different from its endpoints and (2) no three arcs have an interior point in common. The *crossing number* $c(G)$ of G is defined as the minimum number of crossings in a drawing of G satisfying the above conditions, where a *crossing* is a common interior point of two arcs. It is easy to show that the minimum number of crossings can always be realized by a drawing such that

(3) no two arcs meet in more than one point (including their endpoints).

We need the following result which is an easy consequence of a weighted version of the Lipton-Tarjan separator theorem for planar graphs [LT].

Theorem 2.1 *Let G be a graph with n vertices of degree $d_1, \ldots, d_n$. Then*

$$b^2(G) \leq (1.58)^2 \left(16c(G) + \sum_{i=1}^{n} d_i^2\right),$$

where $b(G)$ and $c(G)$ denote the bisection width and the crossing number of G, respectively.

Proof: Let H be a plane graph on the vertex set $V(H) = \{v_1, \ldots, v_N\}$ such that each vertex has a non-negative weight $w(v_i)$ and $\sum_{i=1}^{N} w(v_i) = 1$. Let $d(v_i)$ denote the degree of v_i in H. It was shown by Gazit and Miller [GM] that, by the removal of at most

$$1.58 \left(\sum_{i=1}^{N} d^2(v_i)\right)^{1/2}$$

edges, H can be separated into two disjoint subgraphs H_1 and H_2 such that

$$\sum_{v_i \in V(H_1)} w(v_i) \geq \frac{1}{3} \qquad \sum_{v_i \in V(H_2)} w(v_i) \geq \frac{1}{3}.$$

(See also [M] and [DDS].)

Consider now a drawing of G with $c(G)$ crossing pairs of arcs satisfying conditions (1)–(3). Introducing a new vertex at each crossing, we obtain a plane graph H with $N = n + c(G)$ vertices. Assign weight 0 to each new vertex and weights of $1/n$ to all other vertices. The above result implies that, by the deletion of at most

$$1.58 \left(16c(G) + \sum_{i=1}^{n} d_i^2\right)^{1/2}$$

edges, H can be separated into two parts H_1 and H_2 such that both of the sets $V_1 = V(H_1) \cap V(G)$ and $V_2 = V(H_2) \cap V(G)$ have at least $n/3$ elements. Hence,

$$b(G) \leq |E(V_1, V_2)| \leq 1.58 \left(16c(G) + \sum_{i=1}^{n} d_i^2 \right)^{1/2},$$

and the result follows. □

In the special case when every vertex of G is of degree at most 4, Theorem 2.1 was established by Leighton [L] and it proved to be an important tool in VLSI design (see [U]).

3 Geometric graphs

The aim of this section is to prove the following generalization of Theorem 1.1 for curvilinear graphs.

Theorem 3.1 *Let G be a graph with n vertices that has a drawing with Jordan arcs such that no arc passes through any vertex other than its endpoints, no two arcs meet in more than one point, and there are no $k+1$ pairwise crossing arcs ($k \geq 1$). Then*

$$|E(G)| \leq 3n(10 \log_2 n)^{2k-2}.$$

Proof: By double induction on k and n. The assertion is true for $k = 1$ and for all n. It is also true for any $k > 1$ and $n \leq 6 \cdot 10^{2k-2}$, because for these values the above upper bound exceeds $\binom{n}{2}$.

Assume now that we have already proved the theorem for some k and all n, and we want to prove it for $k+1$. Let $n \geq 6 \cdot 10^{2k}$, and suppose that the theorem holds for $k+1$ and for all graphs having fewer than n vertices.

Let G be a graph of n vertices that can be drawn in the plane so that no two edges meet more than once and there are no $k+2$ pairwise crossing edges. For the sake of simplicity, this drawing will also be denoted by $G = (V(G), E(G))$. For any arc $e \in E(G)$, let G_e denote the graph consisting of all arcs that cross e. Clearly, G_e has no $k+1$ pairwise crossing arcs. Thus, by the induction hypothesis,

$$\begin{aligned}
c(G) &\leq \frac{1}{2}\sum_{e\in E(G)} |E(G_e)| \\
&\leq \frac{1}{2}\sum_{e\in E(G)} 3n(10\log_2 n)^{2k-2} \\
&\leq \frac{3}{2}|E(G)|n(10\log_2 n)^{2k-2}.
\end{aligned}$$

Since $\sum_{i=1}^{n} d_i^2 \leq 2|E(G)|n$ holds for every graph G with degrees $d_1,\ldots,d_n$, Theorem 2.1 implies that

$$\begin{aligned}
b(G) &\leq 1.58\left(16c(G)+\sum_{i=1}^{n} d_i^2\right)^{1/2} \\
&\leq 9\sqrt{n|E(G)|}\,(10\log_2 n)^{k-1}.
\end{aligned}$$

Consider a partition of $V(G)$ into two parts V_1 and V_2, each containing at least n/3 vertices, such that the number of edges connecting them is $b(G)$. Let G_1 and G_2 denote the subgraphs of G induced by V_1 and V_2, respectively. Since neither of G_1 or G_2 contains $k+2$ pairwise crossing edges and each of them has fewer than n vertices, we can apply the induction hypothesis to obtain

$$\begin{aligned}
|E(G)| &= |E(G_1)|+|E(G_2)|+b(G) \\
&\leq 3n_1(10\log_2 n_1)^{2k}+3n_2(10\log_2 n_2)^{2k}+b(G),
\end{aligned}$$

where $n_i=|V_i|$ $(i=1,2)$. Combining the last two inequalities we get

$$\begin{aligned}
&|E(G)|-9\sqrt{n}(10\log_2 n)^{k-1}\sqrt{|E(G)|} \\
\leq\ & 3\frac{n}{3}(10\log_2\frac{n}{3})^{2k}+3\frac{2n}{3}(10\log_2\frac{2n}{3})^{2k} \\
\leq\ & 3n(10\log_2 n)^{2k}(1-\frac{k}{\log_2 n}).
\end{aligned}$$

If the left hand side of this inequality is negative, then $|E(G)|\leq 3n(10\log_2 n)^{2k}$ and we are done. Otherwise,

$$f(x)=x-9\sqrt{n}(10\log_2 n)^{k-1}\sqrt{x}$$

is a monotone increasing function of x when $x \geq |E(G)|$. An easy calculation shows that

$$f(3n(10\log_2 n)^{2k}) > 3n(10\log_2 n)^{2k}(1 - \frac{k}{\log_2 n}).$$

Hence,

$$f(|E(G)|) < f(3n(10\log_2 n)^{2k}),$$

which in turn implies that

$$|E(G)| < 3n(10\log_2 n)^{2k},$$

as required. $\Box$

4 Avoiding snakes

In [ASS], Aronov, Seidel and Souvaine constructed two polygonal regions P and Q with vertices $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_n\}$ in clockwise order such that the size of any piecewise linear homeomorphism $f : P \to Q$ with $f(p_i) = q_i$ $(1 \leq i \leq n)$ is at least cn^2 (for an absolute constant $c > 0$). Their ingenious construction heavily relies on some special geometric features of "snakelike" polygons.

Our Theorem 1.2 (stated in the introduction) provides the same lower bound for a modified version of this problem due to J.E. Goodman. The proof given below is purely combinatorial, and avoids the use of "snakes."

Proof of Theorem 1.2: Let T_1 and T_2 be two triangles containing two convex n-gons P and Q in their interiors, respectively. Let $p_{\pi(1)}, \ldots, p_{\pi(n)}$ denote the vertices of P in clockwise order, where π is a permutation of $\{1, \ldots, n\}$ to be specified later. Furthermore, let $q_1, \ldots, q_n$ denote the vertices of Q in clockwise order. Let $f : T_1 \to T_2$ be a piecewise linear homeomorphism with $f(p_i) = q_i$ $(1 \leq i \leq n)$, and fix a triangulation $\mathcal{T}_1$ of T_1 with $|\mathcal{T}_1| = \texttt{size}(f)$ triangles such that f is linear on each of them. By subdividing some members of $\mathcal{T}_1$ if necessary, we obtain a new triangulation $\mathcal{T}_1'$ of T_1 such that each p_i is a vertex of $\mathcal{T}_1'$ and $|\mathcal{T}_1'| \leq |\mathcal{T}_1| + 3n$.

Obviously, f will map $\mathcal{T}_1'$ into an isomorphic triangulation $\mathcal{T}_2'$ of T_2. The image of each segment $p_{\pi_i}p_{\pi(i+1)}$ is a polygonal path connecting $q_{\pi(i)}$ and $q_{\pi(i+1)}$, $(1 \leq i \leq n)$. The collection of these paths together with the segments q_iq_{i+1} is a drawing of the graph $G = G_\pi$ defined by:

$$(*)\quad \begin{aligned} V(G) &= \{q_1,\ldots,q_n\},\\ E(G) &= \{q_iq_{i+1} \mid 1\le i\le n\} \cup \{q_{\pi(i)}q_{\pi(i+1)} \mid 1\le i\le n\}. \end{aligned}$$

Suppose that this drawing has c crossing pairs of arcs. Notice that each crossing must occur between a path $q_{\pi(i)}q_{\pi(i+1)}$ and a segment q_jq_{j+1}. By the convexity of Q, any line can intersect at most two segments q_jq_{j+1}. Hence the total number of subsegments of the concatenation of the polygons $f(p_{\pi(i)}p_{\pi(i+1)})$, $1\le i\le n$, is at least $c/2$. On the other hand, by the convexity of P, each triangle belonging to $\mathcal{T}_1'$ intersects at most two sides of the form $p_{\pi(i)}p_{\pi(i+1)}$. Thus, $|\mathcal{T}_1'|\ge c/4$, which yields that

$$\texttt{size}(f) = |\mathcal{T}_1| \ge |\mathcal{T}_1'| - 3n \ge \frac{c(G)}{4} - 3n,$$

where $c(G)$ stands for the crossing number of G. Applying Theorem 2.1, we obtain that

$$c(G) > \frac{b^2(G)}{40} - 1.$$

Therefore,

$$\texttt{size}(f) \ge \frac{b^2(G)}{160} - 3n - \frac{1}{4}.$$

To complete the proof of Theorem 1.2, it is sufficient to show that for a suitable permutation π the bisection width of the graph $G=G_\pi$ defined by $(*)$ is at least constant times n. We use a counting argument (cf. [AS]). The family of graphs G_π has size $n!$. We bound from above the number of those members of this family whose bisection width is at most k. We will see that for $k\le n/20$ this number is less than $n!$.

Let $b(G_\pi)\le k$. Let (V_1,V_2) be a partition of $V(G_\pi)$ with $|V_1|,|V_2|\ge n/3$ and $E(V_1,V_2)\le k$. Define

$$\begin{aligned} E_1(V_1,V_2) &= \{q_iq_{i+1} \mid 1\le i< n\}\cap E(V_1,V_2),\\ E_2(V_1,V_2) &= \{q_{\pi(i)}q_{\pi(i+1)} \mid 1\le i< n\}\cap E(V_1,V_2). \end{aligned}$$

Since $|E_1(V_1,V_2)|\le k$, the partition (V_1,V_2) should be of a special form. If we delete all elements of $E_1(V_1,V_2)$ from the path $q_1\ldots q_n$, it splits into at most $k+1$ paths (or points) lying alternately in V_1 and in V_2. This yields a $2(k+1)\binom{n}{k}$ upper bound on the number of partitions in question.

The order in which the elements of V_i $(i=1,2)$ occur in the sequence $q_{\pi(1)}\dots q_{\pi(n)}$ can be represented by a function $\sigma_i : \{1,\dots,|V_i|\} \to V_i$ $(i = 1,2)$. For a fixed partition (V_1,V_2), there are at most $|V_1|!$ choices for σ_1 and $|V_2|!$ choices for σ_2. If σ_1 and σ_2 are also fixed, then the number of possible permutations is bounded again by $2(k+1)\binom{n}{k}$. Thus the total number of permutations π for which $b(G_\pi) \le k$ cannot exceed

$$\begin{aligned}\sum_{(V_1,V_2)} |V_1|!|V_2|!2(k+1)\binom{n}{k} &\le \sum_{(V_1,V_2)} n!\binom{n}{n/3}^{-1} 2(k+1)\binom{n}{k}\\ &\le 4(k+1)^2\binom{n}{k}^2\binom{n}{n/3}^{-1} n!,\end{aligned}$$

which is less than $n!$ if $k \le n/20$, and n is sufficiently large. □

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