Decomposition of multiple coverings into many parts

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Abstract

Suppose that the whole plane (or a large region) is monitored by a set S of stationary sensors such that each element $s \in S$ can observe an axis-parallel unit square $R(s)$ centered at s, which is called the *range* of s. Each sensor s is equipped with a battery of unit lifetime. Is it true that if every point of the plane belongs to the range of many sensors, then we can monitor the plane for a long time without running out of power? If S can be partitioned into k parts $S_1, S_2, \ldots, S_k$ such that, for each i, the sensors in S_i together can observe the whole plane, then the plane can be monitored with no interruption for k units of time. Indeed, we can first switch on all sensors belonging to S_1. After these sensors run out of battery, we can switch on all elements of S_2, etc.

We arrive at the following problem. Let $m(k)$ denote the smallest positive integer m such that any m-fold covering of the plane with axis-parallel unit squares splits into at least k coverings. We show that $m(k) = O(k^2)$, and generalize this result to translates of any centrally symmetric convex polygon in the place of squares. From the other direction, we know only that $m(k) \geq \lfloor 4k/3 \rfloor - 1$.

1 Introduction

The notion of multiple packings and coverings was introduced independently by Davenport and László Fejes Tóth. Given a system $\mathcal{R}$ of subsets of an underlying set X, we say that they form a *m-fold packing (covering)* if every point of X belongs to *at most (at least)* m members of $\mathcal{R}$. A 1-fold packing (covering) is simply called a *packing (covering).* Clearly, the union of m packings (coverings) is always an m-fold packing (covering). Today there is a vast literature on this subject [FTG83], [FTK93]. Throughout this paper, we only consider *locally finite* coverings, that is, we assume that no point belongs to infinitely many

Much of the research on multiple packings and coverings has been concentrated on finding the maximum density of an m-fold packing and the minimum density of an m-fold covering of

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the plane or some higher dimensional Euclidean space with congruent copies or translates of a convex body. There are many results suggesting that, at least in not to high dimensions, the most "economical" configurations have strong structural properties: they are very regular, periodic, even lattice-like, and can be decomposed into simpler parts. If, for instance, an m-fold covering splits into k coverings, then its density is at least k times the minimum density of a covering. But what can be said about "irregular" multiple coverings? Can they be also decomposed into simpler parts? Research in this direction was initiated by László Fejes Tóth in the late 1970s.

Recently, the same problem was raised in a completely different context, in the theory of *large-scale ad hoc sensor networks* [TG02], [LT03], [YZLZ03]. In the so-called Boolean model, the whole plane (or a large region) is monitored by a set S of point-like sensors such that the *range* of each sensor $s \in S$ is a unit disk $R(s)$ centered at s, and each sensor s is equipped with a battery of unit lifetime. Assume further that the family of ranges $\mathcal{R} = \{R(s) : s \in S\}$ is an m-fold covering. If $\mathcal{R}$ splits into k coverings $\mathcal{R}_1, \ldots, \mathcal{R}_k$, the plane can be monitored by the sensors for at least k units of time. Indeed, at time i let us switch on all of those sensors whose ranges belong to $\mathcal{R}_i$ ($1 \leq i \leq k$). The more coverings $\mathcal{R}$ can be decomposed into, the longer uninterrupted service can be guaranteed. This model was communicated to us by Shakhar Smorodinsky [S06].

Given a convex body (region) R in the plane, it is not at all obvious whether there exists a positive integer $m = m(R)$ such that any m-fold covering of the plane with translates of R can be decomposed into *two* coverings! (See [P80].) Even in the special case, when R is a disk, we have only an unpublished manuscript [MP86] (which has never been independently verified), claiming that the statement is true with $m = 33$. As Pach pointed out [AS00], somewhat paradoxically, the difficulty is caused by very heavily covered points. If all points of the plane are covered by congruent disks at least m times and at most $O(2^{m/2})$ times, then it easily follows from the Lovász Local Lemma [EL75] that the arrangement splits into two coverings.

It was shown in [P86] that for any centrally symmetric convex polygonal region R in the plane, there exists a constant $m = m(R)$ satisfying the above condition. The proof has been extended by Tardos and Tóth [TaT06] to the case when R is a triangle. Note that, by simply approximating the disk with centrally symmetric polygons R_n, one cannot deduce the analogous statement for unit disks, because the values $m(R_n)$ may tend to infinity as $n \to \infty$.

For the applications mentioned above, we need much stronger results. Rather than splitting an arrangement into just *two* coverings, we need to decompose it into a large number k of coverings. It was proved in [P86] that for any centrally symmetric convex polygonal region R there exists $\varepsilon = \varepsilon(R) > 0$ such that every m-fold covering of the plane with translates of R can be split into a *covering* and an $\lfloor \varepsilon m \rfloor$-fold covering. Iterating this statement $k - 1$ times, we obtain that for any positive integer k, there exists a constant $m = m(R, k)$ such that any m-fold covering of the plane with translates of R splits into k coverings. The only problem is that the function $m(R, k)$ is huge, it grows exponentially in k.

The aim of this note is to give a quadratic upper bound on this quantity. Our proof will be algorithmic.

Theorem 1. *For any centrally symmetric open convex polygonal region R in the plane, there is a constant $c(R)$ such that every $c(R)k^2$-fold covering of the plane with translates of R can be decomposed into k coverings.*

We believe that the bound in Theorem 1 is far from being optimal. Our best lower bound is linear in k.

Theorem 2. *For any centrally symmetric open convex polygonal region R, there is a $(\lfloor 4k/3 \rfloor - 1)$-fold covering of the plane with translates of R that cannot be decomposed into k coverings.*

2 Preliminaries

In this section, we reformulate Theorem 1 in a dual form and introduce a few notions and notations necessary for the proof. For more details, the interested reader is encouraged to consult [P86], where most of these definitions have originally appeared.

In the sequel, let R denote a fixed open convex polygonal region, centrally symmetric about the origin 0. For any set Q and for any two points $r, s \in \mathbb{R}^2$, let $Q(rs)$ stand for the translate of Q through the vector $\overrightarrow{rs}$. If $r = 0$, for simplicity we write $Q(s)$ for $Q(0s)$. In particular, $R(s)$ denotes a translate of the region R, centered at s.

Let S be a *locally finite* set of points in the plane, that is, suppose that S has no (finite) point of density. From now on, also assume, for simplicity that S is in *general position* with respect to R, in the sense that no line connecting two elements of S is parallel to any side of R. Obviously, $\{R(s) : s \in S\}$ is an m-fold covering of the plane if and only if S has the property that

$$|R(y) \cap S| \geq m \quad \text{for all } y \in \mathbb{R}^2.$$

Thus, Theorem 1 can be rephrased in the following slightly stronger form.

Theorem 2.1. *For any centrally symmetric open convex polygonal region R in the plane, there is a constant $c(R)$ satisfying the following condition. For any $k \geq 2$, the elements of every locally finite set S in the plane can be colored by k colors so that any translate of R that covers at least $c(R)k^2$ points in S contains at least one point of each color.*

Let $\varepsilon = \varepsilon(R)$ be a small positive number such that any square of side ε intersects at most two consecutive sides of R. Partition the plane into squares (cells) of sides ε so that every element of S lies in the interior of a cell. If a translate of R covers at least $c(R)k^2$ points in S, then at least $\frac{c(R)\varepsilon^2}{9\mathrm{diam}^2(R)}k^2$ of them belong to the same cell. Therefore, in order to establish Theorem 2.1, and hence Theorem 1, it is sufficient to prove

Theorem 2.2. *For any centrally symmetric open convex polygonal region R in the plane, there is a constant $c'(R)$ satisfying the following condition. For any $k \geq 2$, the elements of every finite* *set S in a square of side $\varepsilon(R)$ can be colored by k colors so that any translate of R that covers at least $c'(R)k^2$ points of S contains at least one point of each color.*

Denote the vertices of R by $v_1, v_2, \ldots, v_{2n}$, in counterclockwise order. For any i $(1 \le i \le 2n)$, let W_i denote the open convex *wedge* whose apex is at the origin and whose boundary rays are parallel to the vectors $\overrightarrow{v_i v_{i+1}}$ and $\overrightarrow{v_i v_{i-1}}$. Since the set S in Theorem 2.2 lies in a very small square, using the above notation, any translate R' of R satisfies

$$R' \cap S = W_i(x) \cap S \quad \text{for some } 1 \le i \le 2n \text{ and for some } x \in \mathbb{R}^2.$$

In other words, the intersection of S with R' is the same as the intersection of S with a suitable translate $W_i(x) = W_i(0x)$ of some wedge W_i.

Definition 2.3. *The set of all points $s \in S$ for which there exists an i $(1 \le i \le 2n)$ such that the wedge $W_i(s)$ contains no point of S in its interior is called the* boundary *of S and is denoted by* $\mathrm{Bd}(S)$*. A boundary point $s \in \mathrm{Bd}(S)$ is said to be of* type i*, if $W_i(s)$ is empty. Let*

$$\textsc{Type}(s) = \{\ i\ :\ s \text{ is of type } i\ \}.$$

Define a directed graph G on the boundary points of S, as follows. Connect any pair of points $u, v \in \mathrm{Bd}(S)$ by a directed edge (segment) $\overrightarrow{uv} \in E(G)$ if and only if there exist i $(1 \le i \le 2n)$ and $x \in \mathbb{R}^2$ such that $\overrightarrow{xu}$ is parallel to $\overrightarrow{v_i v_{i+1}}$ and $\overrightarrow{xv}$ is parallel to $\overrightarrow{v_i v_{i-1}}$ (with the same orientations), and $W_i(x)$ contains no point of S. By definition, $W_i(x)$ is an open region (wedge), and the points u and v lie on its boundary. In this case, we say that the *type* of the edge $\overrightarrow{uv} \in E(G)$ is i, or, in short, $\textsc{Type}(\overrightarrow{uv}) = i$. Note that the type of every directed edge $\overrightarrow{uv}$ is uniquely determined and is contained in the set $\textsc{Type}(u) \cap \textsc{Type}(v)$. It is possible that the same segment occurs as an edge twice, with opposite orientations. In this case, we have $\textsc{Type}(\overrightarrow{uv}) = i$ and $\textsc{Type}(\overrightarrow{vu}) = i + n$, for some i. Here and everywhere in the sequel, the indices are taken mod $2n$. G is called the *boundary graph* of S. Two boundary points are *neighbors* if they are neighbors in the graph G.

The following simple structural properties of the graph G were established in [P86].

Lemma 2.4.

(i) The edges of a given type form a simple directed polygonal path, which may be empty.

(ii) The edges of G form a directed closed polygonal curve Π that does not cross itself, but may touch itself at several points. Its vertices, the elements of $\mathrm{Bd}(S)$, can be listed in cyclic order as

$$b_{1,0}, \ldots, b_{1,t(1)} = b_{2,0}, \ldots, b_{2,t(2)} = b_{3,0}, \ldots, b_{n,0}, \ldots, b_{2n,t(2n)} = b_{1,0},$$

where the edges of type i form the interval $b_{i,0}, \ldots, b_{i,t(i)}$. We have $i, i-1 \in \textsc{Type}(b_{i,0})$ and $i, i+1 \in \textsc{Type}(b_{i,t(i)})$.

(iii) In this sequence, every boundary point is listed at most twice. If a point $b \in \mathrm{Bd}(S)$ is listed twice, then $\textsc{Type}(b) = \{i, i+n\}$, for some i. We call such a point singular.

(iv) For any $1 \le i \le 2n$ and $x \in \mathbb{R}^2$, the wedge $W_i(x)$ intersects Π in at most two intervals.

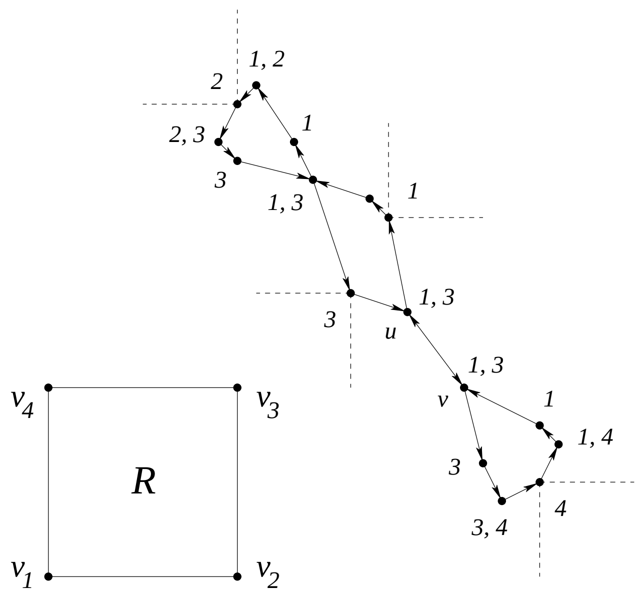

Figure 1: The directed graph G on $\mathrm{Bd}(S)$, where R is the unit square. For each vertex, its TYPE is indicated. Edge uv appears in both directions.

Concerning singular points, in addition to Lemma 2.4 (iii), it is easy to verify

Lemma 2.5.

(i) There is an integer i $(1 \leq i \leq n)$ such that $\mathrm{TYPE}(b) = \{i, i+n\}$, *for every singular point* $b \in \mathrm{Bd}(S)$.

(ii) Let i be the same as in part (i). Both sequences $b_{i,0}, \ldots, b_{i,t(i)}$ and $b_{n+i,0}, \ldots, b_{n+i,t(n+i)}$ contain all singular points, in opposite orders.

3 The coloring algorithm: Proof of Theorem 2.2

The colors used by our algorithm will be denoted by $1, 2, \ldots, k$.

First, we define an auxiliary coloring procedure for any sequence $a_1, \ldots, a_m$ with k colors, where one of the colors i $(1 \leq i \leq k)$ is distinguished. We call this coloring a *periodic* coloring of the sequence with the *special* color i.

PERIODIC-COLOR$(a_1, \ldots, a_m; i)$

For each j $(1 \leq j \leq m)$, color a_j with the special color i if j is odd, *and with color* $1 + (j/2) \pmod{k}$, *if j is* even.

Let S denote the same set of points in a square of side $\varepsilon(R)$, and let

$$b_{1,0}, \ldots, b_{1,t(1)} = b_{2,0}, \ldots, b_{2,t(2)} = b_{3,0}, \ldots, b_{n,0}, \ldots, b_{2n,t(2n)} = b_{1,0}, \tag{1}$$

be the cyclic order of the elements of $\mathrm{Bd}(S)$, as in Lemma 2.4 (ii).

Definition 3.1. *For any positive integer r, a boundary point $b \in \mathrm{Bd}(S)$ is called* r-rich *if there exist j ($1 \leq j \leq 2n$) and $x \in \mathbb{R}^2$ such that the wedge $W_j(x)$ contains more than r elements of S, but $W_j(x) \cap \mathrm{Bd}(S) = b$. Clearly, we have $j \in$ Type(b).*

It is easy to see that a singular point cannot be r-rich for any $r > 1$.

Given S and two integer parameters $i, r > 0$, next we color the boundary of S with k colors, using the following procedure that will be used in our main algorithm as a subroutine.

Color-Boundary(S, i, r)

Step 1. *Color all r-rich vertices of $\mathrm{Bd}(S)$ with color i.*

Step 2. *By Lemma 2.5, we may suppose without loss of generality that all singular boundary points have type $\{1, n+1\}$. Let $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_t$ be the singular (and, hence, non-rich) boundary points, listed in the order as they appear in the sequence $b_{1,0}, \ldots, b_{1,t(1)}$, the initial interval of the list (1). Color them using the algorithm* Periodic-Color*$(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_t; i)$.*

Step 3. *Color all neighbors of every singular boundary point with color i.*

Step 4. *Let $b_1, b_2, \ldots, b_m$ denote the (linear) sequence of uncolored points in the cyclic order (1), starting at the point $b_{1,0}$. Color them using* Periodic-Color*$(b_1, b_2, \ldots, b_m; i)$.*

An important property of this algorithm is the following.

Lemma 3.2. *Among any two consecutive points of the boundary of S in the cyclic order (1), at least one receives color i by the algorithm* Color-Boundary*(S, i, r).* □

Now we can define our main coloring procedure. Let S be the set of points in a square of side $\varepsilon(R)$.

Color-Set(S, k)

Step 0. *Set $i = 1$, $S_1 = S$.*

Step i. *If $S_i = \emptyset$, then* Stop*. Otherwise, apply* Color-Boundary*$(S_i, i, 18k^2 - 18ki)$ to color the set $B_i = \mathrm{Bd}(S_i)$ of all boundary vertices of S_i.*

If $i = k$ then color arbitrarily all uncolored points and Stop*. Otherwise, let $S_{i+1} = S_i \setminus B_i$ and let $i = i + 1$.*

When algorithm Color-Set(S, k) terminates, every point of S is colored by one of the colors $\{1, 2, \ldots, k\}$.

Fix now a wedge $W_j(x)$ with $|W_j(x) \cap S| \geq 18k^2$. To establish Theorem 2.2, we have to show that $W_j(x) \cap S$ contains points of all k colors.

Lemma 3.3. *Suppose that for some i ($1 \leq i \leq k$) and for some wedge $W_j(x)$ we have $|W_j(x) \cap B_i| \geq 18k$. According two Lemma 2.4 (iv), the set $W_j(x) \cap B_i$ is the union of at most two intervals in the counterclockwise cyclic order of boundary points of S_i; denote them by $b_1, b_2, \ldots, b_s$ and $b'_1, b'_2, \ldots, b'_t$. Then at least one of the following two conditions is satisfied:*

(i) At least one element of at least one of the "truncated" intervals $b_2, \ldots, b_{s-1}$, $b'_2, \ldots, b'_{t-1}$, stripped of its endpoints is $(18k^2 - 18ki)$-rich.

(ii) The set $W_j(x) \cap B_i$ contains points of all k colors.

Proof. Suppose that (i) does not hold, that is, none of the elements of the sets $I_1 = \{b_2, \ldots, b_{s-1}\}$ and $I_2 = \{b'_2, \ldots, b'_{t-1}\}$ is $(18k^2 - 18ki)$-rich. (Note that I_1 and I_2 are not necessarily disjoint.)

If $W_j(x) \cap B_i$ contains at least $2k$ singular boundary points, then, by Lemma 2.5, it also contains $2k$ *consecutive* singular boundary points. Since we applied algorithm Periodic-Color to color these points, all k colors must occur among them.

If $W_j(x) \cap B_i$ has at most $2k - 1$ singular boundary points, then consider the set B of all points $b \in W_j(x) \cap B_i$ such that

1. $b \neq b_1, b_s, b'_1, b'_t$,
2. b is not a singular boundary point,
3. b is not a neighbor of a singular boundary point.

Since each singular boundary point has at most four neighbors, we have $|B| \geq |W_j(x) \cap B_i| - 5(2k-1) > 8k$. Therefore, at least one of the sets $B \cap I_1$ and $B \cap I_2$ has at least $4k$ elements.

Suppose without loss of generality that $|B \cap I_1| \geq 4k$. Consider now the *linear* sequence $b_1, b_2, \ldots, b_m$ of uncolored points in the cyclic order (1), starting at the point $b_{1,0}$, in Step 3 of Color-Boundary$(S_i, i, 18k^2 - 18ki)$. The elements of $B \cap I_1$ are consecutive in the *cyclic* order of $B \cap B_i$. Hence, at least half of them, that is, at least $2k$ points, are also consecutive in the linear sequence $b_1, b_2, \ldots, b_m$. These points will receive all k colors, and condition (ii) is satisfied. □

Lemma 3.4. *Suppose that $|W_j(x) \cap S| \geq 18k^2$, and that there is no i ($1 \leq i \leq k$) such that $W_j(x) \cap B_i$ contains points of all k colors. Then we have $|W_j \cap S_i| \geq 18k^2 - 18k(i-1)$, for every i ($1 \leq i \leq k$).*

Proof. The proof is by induction on i. The statement obviously holds for $i = 1$. Assuming that we have already verified the assertion for some $1 \leq i < k$, we want to prove it for $i + 1$.

Since $W_j(x) \cap B_i$ does not contain points of all k colors, there are only two possibilities:

Case A: $|W_j(x) \cap B_i| < 18k$. In this case, we have

$$|W_j(x) \cap S_{i+1}| = |W_j(x) \cap S_i| - |W_j(x) \cap B_i| \geq 18k^2 - 18k(i-1) - 18k = 18k^2 - 18ki.$$

Case B: $|W_j(x) \cap B_i| \geq 18k$. Then, by Lemma 3.3, at least one of the truncated intervals of $W_j(x) \cap B_i$ has an $(18k^2 - 18ki)$-rich point b. According to Definition 3.1, this means that there is a wedge $W_t(y)$ such that $|W_t(y) \cap S_i| > 18k^2 - 18ki$ but $W_t(y) \cap B_i = b$. Thus, we have

$$|W_t(y) \cap S_{i+1}| = |W_t(y) \cap S_i| - |W_t(y) \cap B_i| \geq 18k^2 - 18ki.$$

It is easy to see that in this case $W_j(x) \cap S_{i+1} \supset W_t(y) \cap S_{i+1}$. Hence,

$$|W_j(x) \cap S_{i+1}| \geq 18k^2 - 18ki,$$

as required. □

Now we are in a position to complete the proof of Theorem 2.2, that is, to prove that $W_j(x) \cap S$ contains points of all k colors, provided that $|W_j(x) \cap S| \geq 18k^2$. If there exists an i $(1 \leq i \leq k)$ such that $W_j(x) \cap B_i$ contains points of all colors, we are done. Otherwise, by Lemma 3.4, we have $|W_j \cap S_i| \geq 18k^2 - 18k(i-1) > 0$, for every i $(1 \leq i \leq k)$. Consequently, the set $W_j \cap B_i$ is not empty, for $1 \leq i \leq k$.

If $W_j \cap B_i$ consists of a single point, then this point is r-rich with $r \geq 18k^2 - 18k(i-1) - 1 > 18k^2 - 18ki$, and it receives color i. If $W_j \cap B_i$ has more than one point, then by Lemma 3.2, at least one of its elements must get color i.

Summarizing, for every i $(1 \leq i \leq k)$, Color-Set(S, k) colors at least one element of $W_j(x) \cap S$ with color i. This completes the proof of Theorem 2.2.

4 Construction

As explained at the beginning of Section 2, Theorem 2 can be rephrased in the following equivalent (dual) form.

Theorem 4.1. *For any centrally symmetric open convex polygonal region R in the plane, there exists a locally finite set $S \subset \mathbb{R}^2$ with the following property. Every translate of R covers at least $\lfloor 4k/3 \rfloor - 1$ elements of S, and for any k-coloring of S, one can find a translate R' of R that does not contain points of all colors.*

First, we prove a somewhat weaker statement.

Lemma 4.2. *For any centrally symmetric open convex polygonal region R in the plane and for any $0 < \varepsilon < 1$, there exists a finite set S whose diameter is at most 2ε and which satisfies the following condition. For every k-coloring of S, one can find a translate R' of R such that $|R' \cap S| \geq \lfloor 4k/3 \rfloor - 1$ and R' does not contain points of all colors.*

Proof of Lemma 4.2. As before, let $v_1, v_2, \ldots, v_{2n}$ denote the vertices of R, in counterclockwise order. By applying a suitable linear transformation, if necessary, we can assume that v_1v_2 is horizontal, v_2v_3 is vertical, and the length of each side of R is at least 3. Assume, for simplicity, that k is divisible by 3, and let $\ell = 2k/3$. Let

$$S_1 = \{\varepsilon(1/i, 1/i) : 1 \le i \le \ell\},\ \ S_2 = \{\varepsilon(-1/i, 1/i) : 1 \le i \le \ell\},\ \ S_3 = \{\varepsilon(0, -1/i) : 1 \le i \le \ell-1\},$$

and set $S := S_1 \cup S_2 \cup S_3$. To satisfy the condition that the elements of S are in general position, slightly perturb the coordinates of the points, without changing the notation.

Consider now a coloring of S with k colors. Denote the set of colors missing from S_i by C_i $(i = 1, 2, 3)$. We have $|C_1|, |C_2| \ge k - \ell = k/3$ and $|C_3| \ge k - \ell + 1 > k/3$. Therefore, C_1, C_2, and C_3 cannot be pairwise disjoint, which means that at least one color is missing from at least one of the sets $S_1 \cup S_2, S_1 \cup S_3$, and $S_2 \cup S_3$. Notice that each of the sets $S_i \cup S_j$ has at least $2\ell - 1 = (4k/3) - 1$ elements, and for each of them there is a translate R_{ij} of R with $R_{ij} \cap S = S_i \cup S_j$ $(1 \le i < j \le 3)$. It follows that there is a translate R_{ij} of R such that $|R_{ij} \cap S| \ge (4k/3) - 1$ and R_{ij} does not contain points of all k colors. This proves the lemma. $\square$

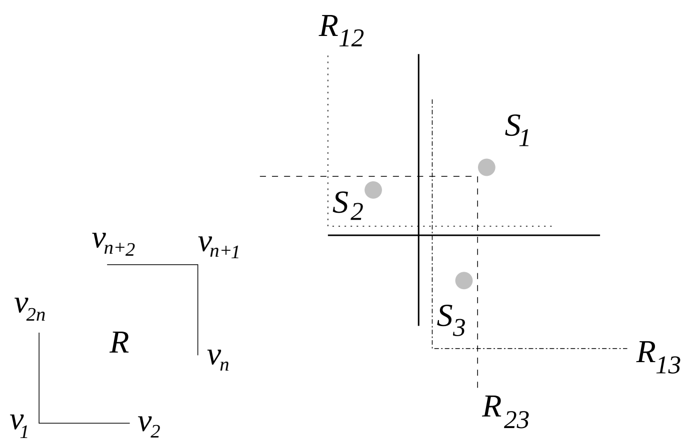

Figure 2: The construction. $|S_1| = \ell - 1 = 2k/3 - 1$, $|S_2| = |S_3| = \ell = 2k/3$.

To establish Theorem 4.1, and hence Theorem 2, it is enough to notice that, for a proper choice of the translates R_{ij} $(1 \le i < j \le 3)$, if we fill $\mathbb{R}^2$ by a sufficiently dense mesh S^*, the set $S^* \cup S$ will meet the requirements stated in Theorem 4.1 for the set S. The details are left to the reader.

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