How many ways can one draw a graph?

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Dedicated to Miklós Simonovits on his sixtieth birthday

Abstract

Using results from extremal graph theory, we determine the asymptotic number of *string graphs* with n vertices, i.e., graphs that can be obtained as the intersection graph of a system of continuous arcs in the plane. The number becomes much smaller, for any fixed d, if we restrict our attention to systems of arcs, any two of which cross at most d times. As an application, we estimate the number of different drawings of the complete graph K_n with n vertices under various side conditions.

1 Introduction

Given a simple graph G, is it possible to represent its vertices by simply connected regions in the plane so that two regions overlap if and only if the corresponding two vertices are adjacent? In other words, is G isomorphic to the *intersection graph* of a set of simply connected regions in the plane? This deceptively simple extension of propositional logic and its generalizations are often referred to in the literature as *topological inference problems* [CGP98a], [CGP98b],[CHK99]. They have proved to be relevant in the area of geographic information systems [E93], [EF91] and in graph drawing [DETT99]. In spite of many efforts [K91a], [K98] (and false claims [SP92], [ES93]), until very recently no algorithm was known for their solution. Two years ago, we showed [PT02] that the problem is *decidable.* Shortly after a more elegant proof was found by Schaefer and Stefankovič [SS01a], who went on to prove that the question is in NP [SS01b].

Since each element of a finite system of regions in the plane can be replaced by a simple continuous arc ("string") lying in its interior so that the intersection pattern of these arcs is the same as that of the original regions, it is enough to restrict our attention to *string graphs,* i.e., to intersection graphs of planar curves. As far as we know, these graphs were first studied in 1959 by S. Benzer [B59], who investigated the topology of genetic structures. Somewhat later they were also considered by F. W. Sinden [S66] in Bell Labs, who was interested in electrical networks realizable by printed circuits. Sinden collaborated with R. L. Graham, who popularized the notion among combinatorists at a conference in Keszthely (Hungary), in 1976 [G78]. Soon after G. Ehrlich, S. Even, and R. E. Tarjan [EET76] studied string graphs (see also

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[K83] and [EPL72] for a special case). The aim of this paper is to estimate the *number* of different string graphs on n vertices.

To formulate our main result precisely, we have to agree on the terminology. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *string representation* of G is an assignment of simple continuous arcs to the elements of $V(G)$ such that two arcs cross each other if and only if the corresponding vertices of G are adjacent. Graph G is a *string graph* if it has a string representation. We assume that any two arcs share only finitely many points and that at each common point the arcs *properly cross*, i.e., one arc passes from one side of the other arc to the other side. An intersection point of two arcs is called a *crossing.*

For any $d > 0$, graph G is a *string graph of rank d* if it has a string representation with the property that any two strings have at most d crossings.

A class $\mathcal{P}$ of labeled graphs, which is closed under isomorphism, is said to be a *property.* A property $\mathcal{P}$ is called *hereditary* if every induced subgraph of every member of $\mathcal{P}$ belongs to $\mathcal{P}$. Let $\mathcal{P}^n$ denote the set of all (labeled) graphs on the vertex set $\{1, 2, \ldots, n\}$ that belong to $\mathcal{P}$. In the combinatorics literature, the function $|\mathcal{P}^n| \leq 2^{\binom{n}{2}}$ is often called the *speed* of property $\mathcal{P}$, and there are several well known estimates on its growth rate as n increases.

Let $\mathcal{S}$ and $\mathcal{S}_d$ denote the classes of all string graphs and all string graphs of rank d, respectively. Clearly, these are hereditary properties and we have $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots \subseteq \mathcal{S}$. Our first goal is to estimate their speeds.

Theorem 1. *For the number $|\mathcal{S}^n|$ of all string graphs on n labeled vertices, we have*

$$2^{\frac{3}{4}\binom{n}{2}} \leq |\mathcal{S}^n| \leq 2^{\left(\frac{3}{4}+o(1)\right)\binom{n}{2}}.$$

Theorem 2. *For any $d > 0$, the number $|\mathcal{S}_d^n|$ of all string graphs of rank d satisfies $|\mathcal{S}_d^n| \leq 2^{o(n^2)}$.*

We do not have any better lower bound on $|\mathcal{S}_d^n|$ than $2^{\Omega(n \log n)}$, which follows from the fact that the vertex set has this many different permutations.

A *drawing* of a graph is a mapping f which assigns to each vertex of G a distinct point in the plane and to each edge uv a continuous arc between $f(u)$ and $f(v)$, not passing through the image of any other vertex. For simplicity, the point assigned to a vertex is also called a *vertex* and an arc assigned to an edge is also called an *edge* of the drawing, and, if this leads to no confusion, it is also denoted by uv. We assume that (a) two edges have only finitely many points in common, and (b) if two edges share an interior point p, then they properly cross at p. Two drawings of G are said to be *essentially equivalent* if the set of crossing pairs of edges is the same in the two drawings. Otherwise, they are *essentially different.*

Let $\Delta(n)$ and $\bar{\Delta}(n)$ denote the number of essentially different drawings and essentially different straight-line drawings, resp., of the complete graph K_n with n vertices. For any $d > 0$, let $\Delta_d(n)$ denote the number of drawings with the property that any two edges have at most d points in common. Clearly, we have

$$\bar{\Delta}(n) \leq \Delta_1(n) \leq \Delta_2(n) \leq \Delta_3(n) \leq \ldots \leq \Delta(n),$$

for every n.

In Sections 2 and 3, we review the extremal graph theoretic tools used in this paper and establish Theorem 1, respectively. In Section 4 we prove Theorem 2, first in the special case $d = 1$, and then we extend the argument to the general case.

In Section 5, we deduce the following estimates.

Theorem 3. *For the number of essentially different drawings of* K_n *under various restrictions, we have*
(i) $2^{\Omega(n \log n)} \leq \bar{\Delta}(n) \leq 2^{O(n \log n)}$;
(ii) $2^{\Omega(n^2)} \leq \Delta_1(n) \leq 2^{O(n^2 \log n)}$;
(iii) $2^{\Omega(n^2 \log n)} \leq \Delta_d(n) \leq 2^{o(n^4)}$, *for any fixed* $d \geq 2$*;*
(iv) $2^{\Omega(n^4)} \leq \Delta(n) \leq 2^{O(n^4)}$.

2 Tools from extremal graph theory

One of the central questions in extremal graph theory [B78] is the following. Given a graph H, what is the maximum number of edges that a graph of n vertices can have if it does not contain H as a (not necessarily induced) subgraph? This quantity is usually denoted by $\mathrm{ex}(n, H)$.

Obviously, the property that a graph is H-free, is hereditary. Let $\mathrm{Forb}(n, H)$ denote the *speed* of this property, i.e., the number of graphs on n labeled vertices that do not contain H as a subgraph. It turns out that the growth rate of these functions crucially depends on the *chromatic number* $\chi(H)$ of H.

Theorem 2.1. (Erdős-Stone [ES46], Erdős-Simonovits [ES66]) *For any graph* H*, we have*

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \frac{n^2}{2} + o(n^2).$$

Theorem 2.2. (Erdős-Frankl-Rödl [EFR86]) *For any graph* H*, we have*

$$\mathrm{Forb}(n, H) = 2^{(1+o(1))\mathrm{ex}(n,H)}.$$

If we want to establish analogous results for graphs containing no *induced* subgraph isomorphic to H, then the first difficulty we have to face is the following: unless H is a complete graph, the maximum number of edges that a graph of n vertices can have without containing an induced copy of H is $\binom{n}{2}$. Thus, Theorem 2.1 does not have a direct analogue. Nevertheless, set

$$\mathrm{ex}^*(n, H) := \left(1 - \frac{1}{\tau(H) - 1}\right) \frac{n^2}{2} + o(n^2),$$

where the relevant quantity, $\tau(H)$, taking the place of the chromatic number is defined as follows.

We say that H is (r, s)*-colorable* for some $0 \leq s \leq r$ if there is an r-coloring of the vertex set $V(H)$, in which the first s color classes are *cliques* (i.e., induce complete subgraphs) and the remaining $r - s$ color classes are *independent sets* (i.e., induce empty subgraphs). Let $\mathcal{C}(r, s)$ denote the class of all (r, s)-colorable graphs, i.e.,

$$\mathcal{C}(r, s) = \{H \ : \ H \text{ is } (r, s)\text{–colorable}\}.$$

Let $\tau(H)$ be the minimum integer r such that H is (r, s)-colorable for all $0 \leq s \leq r$. Clearly, we have $\tau(H) \geq \chi(H)$, for every H.

Let $\mathrm{Forb}^*(n, H)$ stand for the number of graphs on n labeled vertices which does contain H as an induced subgraph.

Theorem 2.3. (Prömel-Steger [PS92]) *For any graph H, we have*

$$\mathrm{Forb}^*(n,H) = 2^{(1+o(1))\mathrm{ex}^*(n,H)}.$$

Using Szemerédi's Regularity Lemma, Bollobás and Thomason [BT97] generalized this result to any nonempty hereditary graph property $\mathcal{P}$. Define the *coloring number* $r(\mathcal{P})$ of $\mathcal{P}$ as the largest integer r for which there is an s such that all (r,s)-colorable graphs have property $\mathcal{P}$. That is,

$$r(\mathcal{P}) = \max\{r \ : \ \text{there exists } 0 \le s \le r \ \text{ such that } \mathcal{P} \supset \mathcal{C}(r,s)\}.$$

Consequently, for any $0 \le s \le r(\mathcal{P})+1$, there exists an $(r(\mathcal{P})+1, s)$-colorable graph that does not have property $\mathcal{P}$.

In the special case when $\mathcal{P}$ is the property that the graph does not contain any induced subgraph isomorphic to H, we have $r(\mathcal{P}) = \tau(H) - 1$.

Theorem 2.4. (Bollobás-Thomason [BT97]) *Let $\mathcal{P}$ be a nontrivial hereditary property of graphs, and let $\mathcal{P}^n$ denote the set of all graphs in $\mathcal{P}$ on the vertex set $\{1, 2, \ldots n\}$. Then the speed of property $\mathcal{P}$ satisfies*

$$|\mathcal{P}^n| = 2^{\left(1-\frac{1}{r(\mathcal{P})}+o(1)\right)\binom{n}{2}},$$

where $r(\mathcal{P})$ is the coloring number of $\mathcal{P}$.

3 String graphs – Proof of Theorem 1

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Figure 1.
Lower bound construction for the number of string graphs.

We start with the lower bound. Consider four pairwise tangent non-overlapping disks D_i, $1 \le i \le 4$, in the plane (see Fig. 1). Assume for simplicity that n is divisible by 4. The proof for other values of n is analogous. Replace the boundary of each D_i by $n/4$ slightly smaller concentric circles C_{ik}, $1 \le k \le n/4$, running very close to it. Fix a pair (i,j), $1 \le i < j \le 4$. By local deformation of every C_{ik} in a small neighborhood of the point of tangency of D_i and D_j, we can achieve that every C_{ik} has a point lying outside every other C_{ih}, $h \ne k$. For every $1 \le l \le n/4$ and for any predetermined set of indices $K_l \subseteq \{1, 2, \ldots, n/4\}$, we can now slightly modify C_{jl} so that it would intersect a curve C_{ik} if and only if $k \in K_l$. In other words, we can arbitrarily specify the bipartite crossing pattern between the curves C_{ik} and C_{jl}, $1 \le k, l \le n/4$. Repeating the same procedure for every pair (i,j), we can obtain any 4-partite crossing pattern between the 4 classes, each containing $n/4$ curves. Note that every C_{ik} is a closed curve, but deleting any point of it which does not belong to another curve it becomes a string. Thus, the number of essentially different string graphs is at least $2^{\frac{6n^2}{16}} > 2^{\frac{3}{4}\binom{n}{2}}$.

Next, we establish the upper bound. For any $r \ge 2$, let G_r be a graph with vertex set

$$V(G_r) = \{v_{ij} \ : \ 1 \le i, j \le r\}$$

and edge set

$$E(G_r) = \{v_{ij}v_{ik} \,:\, 1 \le i, j, k \le r,\, j \ne k\}\,,$$

where $v_{ij} = v_{ji}$, for every i and j. In other words, the vertices of G_r represent the vertices and the edges of the complete graph K_r, two vertices of G_r being connected if the corresponding two edges of K_r share an endpoint or the corresponding edge and vertex of K_r are incident.

Lemma 3.1. *We have $\tau(G_r) = r$.*

Proof. The vertices $v_{1j}, 1 \le j \le r$ form a clique of size r. Therefore, we have $\tau(G_r) \ge \chi(G_r) \ge r$.

Now we show by induction on r that $\tau(G_r) = r$. This is true for $r = 2$. Let $r > 2$ be fixed and assume $\tau(G_{r-1}) = r - 1$. We have to show that, for any $0 \le s \le r$, the vertices of G_r can be colored by r colors so that s color classes induce cliques and the remaining $r - s$ color classes are independent sets.

For $s = 0$, the following coloring will satisfy the requirements. For any $1 \le k \le r$, color a vertex v_{ij} with color k if and only if $i + j \equiv k \bmod r$. Clearly, each vertex of G_r receives a color and each color class is an independent set.

If $s > 0$, color each vertex of the clique $\{v_{1j} \,:\, 1 \le j \le r\}$ with color 1. The uncolored vertices induce a subgraph isomorphic to G_{r-1}, for which we have $\tau(G_{r-1}) = r - 1$, by the induction hypothesis. So the remaining vertices can be colored by $r - 1$ colors so that $s - 1$ color classes induce cliques and the other $r - s$ are independent sets. □

Lemma 3.2. *G_5 is not a string graph.*

Proof. Suppose that G_5 has a string representation. Continuously contract each of string (arc) representing v_{ii} ($1 \le i \le 5$) to a point p_i, without changing the crossing pattern. For every pair $i \ne j$, consider the portion of the arc representing v_{ij} between the points p_i and p_j. These arcs define a drawing of K_5, in which no two independent edges cross each other. However, K_5 is not a planar graph, hence, by a well known theorem of Hanani and Tutte [Ch34], [T70], no such drawing exists. □

Now we can complete the proof of Theorem 1. By Lemma 3.2, a string graph cannot have an induced subgraph isomorphic to G_5. Thus, in view of Lemma 3.1, Theorem 1 directly follows from Theorem 2.3:

$$|\mathcal{S}^n| \le Forb_n^*(G_5) = 2^{\left(\frac{3}{4}+o(1)\right)\binom{n}{2}}.$$

4 String graphs of a fixed rank – Proof of Theorem 2

In order to show that there are $2^{o(n^2)}$ string graphs of rank d, in view of Theorem 2.4, it is enough to exhibit a $(2,0)$-colorable, a $(2,1)$-colorable, and a $(2,2)$-colorable graph such that none of them is a string graph of rank d.

For simplicity, we first we present the argument in the special case $d = 1$. The proof in the general case, presented in Subsection 4.2, is based on the same ideas, but it is technically more complicated.

Let $H_{3,3}$ denote a graph with vertices u_i, v_j, and w_{ij}, $1 \le i, j \le 3$ and edges $u_iw_{ij}, w_{ij}v_j$, for every i and j. In other words, $H_{3,3}$ is the graph obtained from $K_{3,3}$, the complete bipartite graph with three vertices in its classes, by subdividing each of its edges by an extra vertex.

For any k, let T_k denote a graph with vertices $v_i, (1 \le i \le k)$ and u_I, for every $I \subseteq \{1, 2, \ldots, k\}$. Let v_i and v_j be connected by an edge of T_k, for any $1 \le i < j \le k$, and let v_i be connected to u_I if and only if $i \in I$. Let T'_k denote the graph obtained from T_k by adding the edges u_Iu_J, for every $I \ne J$.

Clearly, $H_{3,3}$ is $(2,0)$-colorable (bipartite), T_k is $(2,1)$-colorable, and T'_k is $(2,2)$-colorable, for every k. Therefore, if $\mathcal{P} = \mathcal{P}(H_{3,3}, T_k, T'_k)$ denotes the property that a graph does not contain $H_{3,3}$, T_k, or T'_k as an induced subgraph, then $\mathcal{P}$ is a *hereditary* property with coloring number $r(\mathcal{P}) = \infty$. Hence, by Theorem 2.4, for the number of graphs on n labeled vertices, satisfying property $\mathcal{P}$, we have $|\mathcal{P}^n| = 2^{o(n^2)}$.

4.1 String graphs of rank 1

It remains to prove the following statement, which implies that $\mathcal{S}_1^n \subseteq \mathcal{P}^n$ if k is large enough.

Lemma 4.1. *A string graph of order* 1 *cannot contain* $H_{3,3}$*,* T_k*, or* T'_k *as an induced subgraph, provided that* k *is sufficiently large.*

Proof. It is well known that a string graph cannot contain $H_{3,3}$ as an induced subgraph (see e.g. [EET76],

Using the notation in the definition of T_k (and T'_k), let v_i, $1 \le i \le k$ and u_I, $I \subseteq \{1, 2, \ldots, k\}$ stand for the vertices of T_k (and T'_k, resp.), and suppose that T_k (and T'_k, resp.) has a string representation in which any two strings cross at most once. For simplicity, we use the same notation for the strings as for the corresponding vertices.

Fix arbitrarily an orientation of each string. For any triple (x, y, z), $1 \le x < y < z \le k$, let $f_{xyz} = 1$ if along v_y the crossing with v_x *follows* the crossing with v_z. Otherwise, set $f_{xyz} = 0$.

By Ramsey's theorem, there exists a "homogeneous" subset $J \subseteq \{1, 2, \ldots, k\}$, $|J| \ge \log\log k$, such that f_{xyz} is constant over all triples (x, y, z), $1 \le x < y < z \le k$, $x, y, z \in J$. We can assume without loss of generality that $J = \{1, 2, \ldots, m\}$, where $m \ge \log\log k$.

For any $1 \le i \le m$, the string v_i crosses all other v_j, $1 \le j \le m$, $i \ne j$ exactly once. Since f_{xiz} is constant over all triples (x, i, z), $1 \le x < i < z \le k$, one can find a non-crossing point on v_i that divides v_i into two parts, $v_i^<$ and $v_i^>$, containing all crossings between v_i and v_x with $x < i$ and between v_i and v_z with $z > i$, respectively. The arcs $v_i^<$ and $v_i^>$ are called the *lower part* and the *upper part* of v_i, respectively.

Construct two 42-uniform hypergraphs, $H^<$ and $H^>$, both on the vertex set $\{1, 2, \ldots, m\}$, as follows. For any $1 \le x_1 < x_2 < \cdots < x_{83} \le m$, there exists a string $u = u_{\{x_1, x_2, \ldots, x_{83}\}}$ that crosses $v_{x_1}, v_{x_2}, \ldots, v_{x_{83}}$, but no other v_j. The string u crosses either the lower or the upper part of each v_{x_i}, so for at least $a \ge 42$ indices $1 \le i \le 83$ it will cross, say, the lower (resp., upper) part. Suppose, for example, that u crosses the lower (resp., upper) parts of $v_{x_1}, v_{x_2}, \ldots, v_{x_a}$ *in this order* along u. Then add the hyperedge $\{x_1, x_2, \ldots, x_{42}\}$ to $H^<$ (resp., to $H^>$). Let u' be the segment of u from its beginning to its crossing with $v_{x_{42}}$. GÉZA MEGJEGYZÉSE: **megnezni** ⟵

Repeating the above procedure for every 83-tuple $1 \le x_1 < x_2 < \cdots < x_{83} \le m$, the total number of hyperedges in $H^<$ and $H^>$ with *repetitions* is $\binom{m}{83}$. However, the multiplicity of each hyperedge is at most $\binom{m-42}{41}$. Thus, the total number of *distinct* hyperedges in $H^<$ and $H^>$ is $\Omega(m^{42})$ (i.e., at least constant times m^{42}). Suppose without loss of generality that $H^<$ has $\Omega(m^{42})$ distinct hyperedges.

We can now apply a well known result of Erdős [E65] (see also [B78] and [PA95], p. 151) to conclude that, for any fixed l and sufficiently large m, our hypergraph $H^<$ contains a complete 42-partite, 42-uniform subhypergraph $K^{42}_{l,\ldots,l}$ with l elements in each of its classes. (That is, $K^{42}_{l,\ldots,l}$ has $42l$ vertices, divided into 42 classes of size l, and it consists of all 42-tuples that contain one vertex from each class.)

For simplicity, denote by s_i^j, $1 \le i \le 42$, $1 \le j \le l$ the *lower* parts $v_{x_k}^<$ of the strings v_{x_k} corresponding to the vertices of $K^{42}_{l,\ldots,l}$. By the construction, for each 42-tuple $(j_1, \ldots, j_{42})$, $1 \le j_1, \ldots, j_{42} \le l$, there exists a string $u'_{j_1,\ldots,j_{42}}$ that crosses $s_1^{j_1}, \ldots, s_{42}^{j_{42}}$, but no other string s_i^j.

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Figure 2.
Some of the strings representing a $K^{42}_{3,\dots,3}$.

Color the 42-tuples $(j_1,\dots,j_{42})$ with 42! colors, according to order in which the crossings with $s_1^{j_1},\dots,s_{42}^{j_{42}}$ occur along $u_{j_1,\dots,j_{42}}$. Thus, we can find at least $\Omega(l^{42})$ 42-tuples of the same color (say, white). Suppose without loss of generality that, for each such 42-tuple $(j_1,\dots,j_{42})$, the string $u_{j_1,\dots,j_{42}}$ first crosses $s_1^{j_1}$, then $s_2^{j_2}$,..., and finally $s_{42}^{j_{42}}$. Applying Erdős's result again, if l is sufficiently large, we can find a subhypergraph $K^{42}_{3,\dots,3}\subset K^{42}_{l,\dots,l}$, all of whose 42-tuples are white. Again, we can assume without loss of generality that the strings corresponding to the vertices of $K^{42}_{3,\dots,3}$ are s_i^j, $1\le i\le 42$, $1\le j\le 3$. Recall that each s_i^j is the *lower* part of an original string v_x, therefore, no two s_i^j can cross each other. (Indeed, the intersection of v_x and v_y, $x<y$, must belong to the upper part of v_x and at to the lower part of v_y.)

Summarizing: we have $3\cdot 42=126$ strings s_i^j, $1\le i\le 42$, $1\le j\le 3$, no two of which intersect. Moreover, for each 42-tuple $(j_1,\dots,j_{42})$, $1\le j_1,\dots,j_{42}\le 3$, there is a string $u_{j_1,\dots,j_{42}}$ that intersects the strings $s_1^{j_1},\dots,s_{42}^{j_{42}}$ in this order, and does not intersect any other s_i^j. (See Fig. 2.) We would like to show that there are two different strings of the type $u_{j_1,\dots,j_{42}}$ that cross more than once. First, we give a lower bound for the number of crossings $\text{CR}(u,u)$ between strings of type $u_{j_1,\dots,j_{42}}$.

Let $1\le x\le 41$ be fixed. For any pair y,z, $1\le y,z\le 3$, consider all strings $u_{j_1,\dots,j_{42}}$ with $j_x=y$ and $j_{x+1}=z$, and let $\Gamma_{y,z}$ denote the set of their portions between their intersections with s_x^y and s_{x+1}^z. Clearly, we have $|\Gamma_{y,z}|=3^{40}$. Pick one element from each $\Gamma_{y,z}$, $1\le y,z\le 3$, and notice that at least one pair among these 9 arcs must be crossing, otherwise, together with the strings s_x^1,s_x^2,s_x^3, $s_{x+1}^1,s_{x+1}^2,s_{x+1}^3$, they would give a string representation of $H_{3,3}$, which is impossible (see the first paragraph of this proof). Thus, for a fixed x, the total number of crossings between the elements of $\Gamma_{y,z}$ and $\Gamma_{y',z'}$ over all y,z,y',z', $1\le y,z,y',z'\le 3$, $(y,z)\ne(y',z')$ is at least

$$\frac{\prod_{1\le y,z\le 3}|\Gamma_{y,z}|}{3^{7\cdot 40}}=\frac{3^{9\cdot 40}}{3^{7\cdot 40}}=3^{80}.$$

Here the denominator, $3^{7\cdot 40}$, is the number of 9-tuples of arcs, one from each set $\Gamma_{y,z}$, $1\le y,z\le 3$, in which a crossing pair of arcs is fixed. Repeating this count for every x, $1\le x\le 41$ and noticing that every time we count different crossings, we obtain that

$$\text{CR}(u,u)\ge 41\cdot 3^{80}.$$

On the other hand, the number of strings of type $u_{j_1,\dots,j_{42}}$ is 3^{42}. If any two of them cross at most once, than $\text{CR}(u,u)<3^{84}/2$, which is a contradicts the above inequality. This completes the proof of the lemma. □

4.2 String graphs of larger ranks

In this subsection, we generalize the above proof to string graphs of any fixed rank $d>0$. As before, Theorem 2 is an immediate consequence of the following lemma.

Let $d>0$ be fixed throughout the proof.

Lemma 4.2. *Let d be a fixed positive integer. Then no string graph of order d can contain $H_{3,3}$, T_k, and T'_k as an induced subgraph, provided that $k \ge k_0(d)$.*

Proof. We have already seen that no string graph contains $H_{3,3}$ as an induced subgraph.

As in the beginning of Subsection 4.1, let v_i, $1 \le i \le k$ and u_I, $I \subset \{1,2,\ldots,k\}$ denote the vertices of T_k (and T'_k). Suppose, in order to obtain a contradiction that T_k (and T'_k, resp.) has a string representation in which any two strings cross at most d times, and we fix such a representation. Again, for simplicity, we use the same notation for the strings as for the corresponding vertices, and we arbitrarily fix an orientation for each string.

By repeated application of the theorems of Ramsey and Erdős, used in the previous subsection, we will conclude that, if k is sufficiently large compared to d, then the string representation of T_k (and T'_k) must contain a "large regular" subconfiguration, similar to the one depicted in Figure 2. On the other hand, it will be not difficult to argue that in every such configuration there are two strings that cross more than d times. This contradiction will prove the lemma. (For clearer presentation, we will not make an effort to improve the dependence of k on d by minimizing the number of applications of Ramsey's and Erdős's theorems.)

Any two strings cross at most d times. By Ramsey's theorem, if k is sufficiently large, then we can select an arbitrarily large subset of the v_i's so that any two of them intersect the same number of times. We restrict our attention to such a subset. Slightly abusing the notation, from now on we assume that any two of $v_1, v_2, \ldots, v_k$ cross exactly d times.

To each triple (x,y,z), $1 \le x < y < z \le k$, assign a word $W(x,y,z)$ of length $2d$, consisting of d 0's and d 1's, as follows. Traveling along v_y, write a 0 or a 1 whenever we find a crossing with v_x or v_z, respectively. The number of different words created in this way is at most $\binom{2d}{d}$. Thus, we can apply Ramsey's theorem again (this time for triples) to show that, if k is sufficiently large, there exists an arbitrarily large subset $J \subset \{1,2,\ldots,k\}$ such that we assigned the same word to all triples (x,y,z) from J with $x < y < z$. In the rest of the proof, we only consider the elements of this subset. By a further abuse of notation, we assume that $J = \{1,2,\ldots,k\}$.

For any x, y, $1 \le x \ne y \le m$, let $\kappa_y^x(i)$ denote the i-th intersection of v_x and v_y, in the order of appearance along v_y. In particular, we have that $\kappa_x^y(i) = \kappa_y^x(j)$ for some j, $1 \le j \le d$.

For any y, $1 \le y \le k$, define $v_y^{<}(i)$, the *i-th lower part* (and $v_y^{>}(i)$, the *i-th higher part*) of v_y as the *smallest closed* interval of v_y that contains all intersection points $\kappa_y^x(i)$ with $x < y$ (resp., all intersection points $\kappa_y^z(i)$ with $z > y$. Using the fact that $W(x,y,z)$ is the same for all $1 < x < y < z \le k$, we obtain that $v_y^{<}(i)$ and $v_y^{>}(j)$ are disjoint for any $1 \le i,j \le k$. Moreover, either $v_y^{<}(i)$ comes after $v_y^{>}(j)$ along v_y for all y, or $v_y^{<}(i)$ comes before $v_y^{>}(j)$ along v_y for all y. Note, however, that $v_y^{<}(i)$ and $v_y^{<}(j)$, or $v_y^{>}(i)$ and $v_y^{>}(j)$ may overlap. Deleting from v_y its i-th lower and upper parts, $v_y^{<}(i)$ and $v_y^{>}(i)$, for every $1 \le i \le d$, we are left with at most $D = 2d+1$ open intervals, $v_y^*(i)$, $1 \le i \le D$, where $v_y^*(i)$ is called the *i-th separating part* of v_y.

Let $r := 41d + 1$ and $R := (2d + D)r$. We define $2d + D$ r-uniform hypergraphs, $G^{<}(i)$, $G^{>}(i)$ $(1 \le i \le d)$ and $G^*(i)$ $(1 \le i \le D)$, on the vertex set $\{1,2,\ldots,k\}$, as follows. Fix an R-tuple $(x_1, x_2, \ldots, x_R)$, $1 \le x_1 < x_2 < \cdots < x_R \le k$. By the definition of T_k (and T'_k), there exists a string $u_{\{x_1,x_2,\ldots,x_R\}}$ that crosses $v_{x_1}, v_{x_2}, \ldots, v_{x_R}$ but no other v_x. For every $1 \le i \le R$, consider the *last* crossing between v_{x_i} and $u_{\{x_1,x_2,\ldots,x_R\}}$ along $u_{\{x_1,x_2,\ldots,x_R\}}$. It may belong to the j-th lower part or to the j-th upper part of v_{x_i} for some $1 \le i \le d$, or to one of its at most D separating parts. Since there are at most $2d+D$ possibilities, there are $a \ge r$ strings, $v_{x_{i(1)}}, v_{x_{i(2)}}, \ldots, v_{x_{i(a)}}$, such that their last crossings with $u_{\{x_1,x_2,\ldots,x_R\}}$

all belong to their same parts. Suppose for instance that the last crossings of $v_{x_{i(1)}}, v_{x_{i(2)}}, \ldots, v_{x_{i(a)}}$, with $u_{\{x_1,x_2,\ldots,x_R\}}$ appear *in this order* along $u_{\{x_1,x_2,\ldots,x_R\}}$, and each of them belong to the i-th lower part of the corresponding string. Then we add the r-tuple $(v_{x_{i(1)}}, v_{x_{i(2)}}, \ldots, v_{x_{i(r)}})$ to the hypergraph $G^<(i)$, as a hyperedge. Define $u'_{\{x_1,x_2,\ldots,x_R\}}$ to be the smallest interval of $u_{\{x_1,x_2,\ldots,x_R\}}$, that contains its last crossings with $v_{x_{i(1)}}, v_{x_{i(2)}}, \ldots, v_{x_{i(r)}}$.

The r-uniform hypergraphs $G^>(i)$ and $G^*(i)$ are defined analogously. GÉZA MEGJEGYZÉSE: **megnezni** ←—

Repeating this procedure for every R-tuple, we find that (with multiplicities) the total number of hyperedges of all r-uniform hypergraphs $G^<(i)$, $G^>(i)$, and $G^*(i)$ is $\binom{k}{R}$, and the maximum multiplicity of a hyperedge is $\binom{k-r}{R-r}$. Thus, the total number of *different* hyperedges in $G^<(i)$, $G^>(i)$, and $G^*(i)$ is $\Omega(k^r)$, and at least one of them, $G^<(1)$ (say), has $\Omega(k^r)$ distinct edges. (The other cases can be treated similarly.) Now we can apply Erdős's result [E65], mentioned and used before, to show that, for any large fixed l, $G^<(1)$ contains a complete r-partite r-uniform hypergraph $K^r_{l,\ldots,l}$ with l vertices in its classes, provided that k is sufficiently large. Denote the *first low* parts of the strings that represent the vertices of the i-th class of $K^r_{l,\ldots,l}$ by s_i^j, $1 \le j \le l$ $(1 \le i \le r)$. Clearly, for each r-tuple $(j_1, \ldots, j_r)$, $1 \le j_1, \ldots, j_r \le l$, there is a string $u'_{j_1,\ldots,j_r}$ that intersects all elements of $\{s_1^{j_1}, \ldots, s_r^{j_r}\}$, but no other s_i^j.

Color the r-tuples $(j_1, \ldots, j_r)$ with $r!$ colors, as follows. Traveling along $u_{j_1,\ldots,j_r}$ from beginning to end, record the *last* crossings with the elements of $\{s_t^{j_t} : 1 \le t \le r\}$, and at each such crossing write down the corresponding index t. In this way, we obtain a permutation π of the numbers $1, 2, \ldots, r$. Choose a different color for each permutation, and color the r-tuple $(j_1, \ldots, j_r)$ with the color assigned to π. Assume without generality that the largest color class (red, say) corresponds to the identity permutation. If k (and thus l) is sufficiently large, then, applying Erdős's theorem again, we can argue that $K^r_{l,\ldots,l}$ contains a complete r-partite subhypergraph $K^r_{3,\ldots,3}$ with 3 elements in its classes, all of whose hyperedges are red. We can assume without loss of generality that the strings corresponding to the vertices of the i-th class of $K^r_{3,\ldots,3}$ are s_i^1, s_i^2, and s_i^3 $(1 \le i \le r)$. Recall that each s_i^j belongs to the *first lower* part of an original string v_x. Hence, no two s_i^j's can have a point in common. Indeed, any intersection point of v_x and v_y $(x < y)$ would belong to some higher part of v_x. (The same argument works in the other cases, too, as no two higher and no two separating parts belonging to different v_x's can intersect.)

Summarizing: we have $3r$ pairwise disjoint strings, s_i^j, $1 \le i \le r$, $1 \le j \le 3$. For each r-tuple, $(j_1, \ldots, j_r)$, $1 \le j_1, \ldots, j_r \le 3$, there is a string $u_{j_1,\ldots,j_r}$ that intersects every element of $\{s_1^{j_1}, \ldots, s_r^{j_r}\}$, but no other string s_i^j $(1 \le i \le r, 1 \le j \le 3)$. For every $1 \le i \le r$, let $p^i_{j_1,\ldots,j_r}$ denote the last intersection point of $u_{j_1,\ldots,j_r}$ and $s_i^{j_i}$ along $u_{j_1,\ldots,j_r}$. Then the points $p^1_{j_1,\ldots,j_r}, p^2_{j_1,\ldots,j_r}, \ldots, p^r_{j_1,\ldots,j_r}$ must follow each other in this order along $u_{j_1,\ldots,j_r}$. (See Fig. 2 for the special case $r = 42$.)

It remains to show that there exists a pair of strings of the type $u_{j_1,\ldots,j_r}$ that must cross more than d times. This contradiction will complete the proof of Lemma 4.2 and hence Theorem 2.

As at the end of the proof of Lemma 4.1, we estimate $\text{CR}(u,u)$, the number of crossings among the strings $u_{j_1,\ldots,j_r}$, $1 \le j_1, \ldots, j_r \le 3$ in the above drawing, in two different ways.

Let $1 \le x \le r-1$ be fixed. For any pair y, z, $1 \le y, z \le 3$, consider all strings $u_{j_1,\ldots,j_r}$ with $j_x = y$ and $j_{x+1} = z$, and let $\Gamma_{y,z}$ denote the set of their portions between $p^x_{j_1,\ldots,j_r}$ and $p^{x+1}_{j_1,\ldots,j_r}$. Clearly, we have $|\Gamma_{y,z}| = 3^{r-2}$. Pick one element from each $\Gamma_{y,z}$, $1 \le y, z \le 3$, and observe that at least one pair among these 9 arcs must be crossing. Now we obtain that, for a fixed x, the total number of crossings between the elements of $\Gamma_{y,z}$ and $\Gamma_{y',z'}$ over all y, z, y', z', $1 \leq y, z, y', z' \leq 3$, $(y,z) \neq (y',z')$ is at least

$$\frac{\prod_{1\leq y,z\leq 3} |\Gamma_{y,z}|}{3^{7(r-2)}} = \frac{3^{9(r-2)}}{3^{7(r-2)}} = 3^{2(r-2)}.$$

Repeating this count for every x, $1 \leq x \leq r-1$ and noticing that every time we count different crossings, we obtain that

$$\text{CR}(u,u) \geq (r-1) \cdot 3^{2(r-2)}.$$

On the other hand, the number of strings $u_{j_1,\ldots,j_r}$ is 3^r, any two of them cross at most d times, so that

$$\text{CR}(u,u) \leq d \cdot 3^{2r}/2.$$

Since $r = 41d$, the last two bounds contradict each other, and this completes the proof. □

5 Drawings of complete graphs – Proof of Theorem 3

(i) It is easy to see that the *order type* on the vertices of K_n (i.e., the orientation of its triples) determines the set of crossing pairs of edges, So the upper bound follows from a result of Goodman and Pollack [GP86], that there are at most n^{6n} different order types on n points. On the other hand, we can place the vertices of K_n on a circle, in $(n-1)!$ different cyclic order, and each placement gives a different list of crossing pairs of edges. It is also easy to come up with a list of $n^{\Omega(n)}$ drawings such that by relabelling the vertices of any one of them, we do not obtain a drawing essentially the same as another.

(ii) Suppose n is divisible by 4, and let $v_i = (-1, i), u_j = (1, j)$, and $w_k = (0, k/2)$, for any $1 \leq i, j \leq n/4$ and $1 \leq k \leq n/2$. For every $1 \leq k < n/2$, connect w_k and w_{k+1} by a straight-line segment. Furthermore, connect every v_i to every u_j by a line segment so that each such segment passes through some point w_k. By slightly bending each edge v_iu_j, but keeping its endpoints fixed, we can achieve that it passes either slightly above or slightly below w_{i+j}. At each edge v_iu_j, we have two choices, so there are $2^{n^2/16}$ possibilities. In each drawing, any two edges cross at most once, and different choices give rise to different crossing patterns. (Indeed, v_iu_j passes above w_{i+j} if and only if it crosses the edge $w_{i+j}w_{i+j+1}$.) Finally, one can slightly perturb the vertices so that no three of them would be collinear, and connect the missing pairs by straight-line segments without creating more than one crossing between any pair of edges. Therefore, the number of different crossing patterns is at least $2^{n^2/16}$.

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Figure 3.
The eight combinatorially different drawings of K_4.

As for the upper bound, for a fixed drawing, for each vertex v_i, list the edges incident to v_i in clockwise order around v_i. For every vertex, we have $(n-2)!$ possibilities, so there are $((n-2)!)^n < 2^{n^2 \log n}$ different sets of lists. We claim that this set of lists uniquely determines the crossing pattern. To see this take two edges, v_1v_2 and v_3v_4, and consider the drawing of K_4 induced by these vertices, as a drawing on the *sphere.* Two spherical drawings of K_4 are *combinatorially equivalent* if the corresponding maps are isomorphic. There are 8 combinatorially different drawings of K_4, with the property that any two edges have at most one point in common (see Fig. 3), and these drawings can be distinguished by looking at the cyclic orders of edges incident to a vertex. Hence, the cyclic order of edges at the vertices determines whether v_1v_2 and v_3v_4 cross each other.

(iii) Suppose n is divisible by 3. For $i = 1, 2, \ldots, n/3$, let $v_i = (-1, i)$, $w_i = (0, i)$, and $u_i = (1, i)$. Connect every v_i to every u_j, as follows. Choose a number k, $0 \leq k < n/3$, and connect both v_i and u_j to $(0, k+\varepsilon)$ by a segment. Also connect any two consecutive w_i's by a segment. In the resulting drawing, any two edges have at most two common points, and a different choice for any v_iu_j results a different crossing pattern. Therefore, the number of different crossing patterns is $(n/3)^{n^2/9}$. Clearly, each of these drawings can be extended to a drawing of the complete graph such that still any two have at most two common points. For instance, slightly perturb the points together with the existing edges, so that the points are in general position, and add the missing edges as segments.

For the upper bound, apply Theorem 2 for the edges of K_n regarded as $\binom{n}{2}$ strings.

(iv) Suppose n is even, and let $v_i = (-1, i)$, $u_i = (1, i)$, for $1 \leq i \leq n/2$. For any i, j, $1 \leq i < j \leq n/2$, connect v_i with $(0, ni + j)$ and connect $(0, ni + j)$ with v_j. Now, all vertices v_i and all edges connecting them are on the left side of the line $x = 0$ such that each of the edges has exactly one point on that line, and all these points are different. On the other hand, all vertices u_i, are on the right-hand side of the line $x = 0$. So, for any p, q, $1 \leq p < q \leq n/2$, and for any set $K_{pq} \subseteq \{(i, j) \ : \ 1 \leq i < j \leq n/2\}$, we can draw the edge u_pu_q so that it crosses v_iv_j $(i < j)$ if and only if $(i, j) \in K_{pq}$ (cf. proof of Theorem 1). $\square$

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