Rich Cells In An Arrangement Of Hyperplanes

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1 Introduction

Given an arrangement of n hyperplanes in $\mathbb{R}^d$ we call a cell of the arrangement *rich* if its boundary contains a piece of each of the hyperplanes, i.e. it has n facets, one supported by each hyperplane. Here in sections 2-5 we find bounds for $f(d,n)$ the maximum number of rich cells over all such arrangements, we found $f(2,n)$ precisely and prove the following theorem.

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Theorem *For $n \geq d \geq 3$*

$$f(d,n) = \binom{n}{d-2} + O(n^{d-3})$$

The hyperplanes are in convex position if there is some rich cell in their arrangement. In section 6 we find a 'Carathéodory number' for lines in the plane: we show that a set of lines in the plane lie in convex position, provided every five of the lines are in convex position.

2 A Recurrence Relation

Let $H = \{H_1, H_2, \ldots, H_n\}$ be n hyperplanes in $\mathbb{R}^d$ and consider the arrangement $\mathcal{A}(H)$ of these hyperplanes. Let $f_{\mathcal{A}}(d,n)$ be the number of rich cells in $\mathcal{A}(H)$, we want to determine $f(d,n)$ the maximum number of rich cells over all such arrangements.

Lemma 1 $f(d,n) \leq f(d,n-1) + f(d-1,n-1)$, *where* $n, d \geq 2$

Proof

Let $n, d \geq 2$, consider the contribution of H_n: A rich cell of $\mathcal{A}(H)$ can only occur when H_n cuts a rich cell of $\mathcal{A}(H - H_n)$. H_n can cut such a cell into at most two rich cells of $\mathcal{A}(H)$.

If some hyperplane H_k is parallel to H_n then no region on H_n can act as a facet of two rich cells as H_k lies uniquely in H_n^+ say, and cannot bound a facet in H_n^-. Hence:

$$f_{\mathcal{A}}(d,n) \leq f(d,n-1) \leq f(d,n-1) + f(d-1,n-1)$$

Otherwise, say H_n divides a rich cell C of $\mathcal{A}(H - H_n)$ into two rich cells C_A and C_B of $\mathcal{A}(H)$. Then some region R of H_n is a shared facet of C_A and C_B, R lies in the $(d-1)$-flat H_n, we claim that R has $(n-1)$ facets. If R has less than $(n-1)$ facets then some $H_k \cap H_n$, $(1 \leq k < n)$ does not support R. H_n does not cut the facet of cell C due to H_k. This facet must lie uniquely in one halfspace H_n^+ say, but then H_k can support a facet of only one of the cells C_A or C_B. Thus R has $(n-1)$ facets and is a rich cell in the arrangement of the $(d-2)$-flats $H_k \cap H_n$, $k = 1, \ldots, n-1$ lying in H_n. There can be at most $f(d-1,n-1)$ such regions R in H_n. All other rich cells in $\mathcal{A}(H - H_n)$ can yield at most one rich cell in $\mathcal{A}(H)$.

$$f_{\mathcal{A}}(d,n) \leq f(d,n-1) + f(d-1,n-1)$$

□

3 Boundary Conditions

The results when $d = 1$ are obvious: $f(1, 1) = 2$, $f(1, 2) = 1$, and $f(1, k) = 0$ whenever $k \geq 3$. Also it is clear that $f(d, 1) = 2$ for all d. We could use this as the starting position for the recurrence, but a better bound can be obtained if a few more cases are investigated.

Lemma 2 *Whenever* $k \leq d$, $f(d, k) = 2^k$

Proof

We have observed that $f(1, 1) = 2$ and $f(d, 1) = 2$ for all d, so the result holds for $d = 1$, and when $k = 1$. It follows by induction that $f(d, k) \leq 2^k$, and this bound is realised by an arrangement of $k \leq d$ mutually orthogonal hyperplanes in $\mathbb{R}^d$, which has 2^k rich cells. $\square$

Lemma 3 *When* $d \geq 2$

$$\begin{aligned} f(d, d+1) &= \binom{d+1}{d+1} + \binom{d+1}{d} + \cdots + \binom{d+1}{2} \\ &= 2^{d+1} - d - 2 \end{aligned}$$

Proof

(i) Consider the arrangement $\mathcal{A}$ of $(d+1)$ hyperplanes in general position in $\mathbb{R}^d$. This arrangement has one bounded cell: a simplex; all other cells are unbounded and 'hang from' the faces of this simplex. The only cells in this arrangement that are not rich are the $(d + 1)$ cells that hang from the vertices of the simplex.

$$f_{\mathcal{A}}(d, n) = \binom{d+1}{d+1} + \binom{d+1}{d} + \cdots + \binom{d+1}{2}$$

(ii) If any two of the hyperplanes are parallel then each of the parallel planes cannot divide a cell rich in the remaining d planes into two cells rich in the $(d + 1)$ planes. So in this case there can be no more than $f(d, d - 1)$ rich cells and $f(d, d - 1) = 2^{d-1} \leq 2^{d+1} - d - 2$ whenever $d \geq 2$.

(iii) In $\mathbb{R}^2$ if three lines have a point in common then there are no rich cells in the arrangement.

In $\mathbb{R}^3$: If four planes have a line in common then the arrangement is equivalent to four lines with a point in common in two dimensions, and has no rich cells. If four planes have exactly one point in common, the arrangement of three of the planes has at most 2^3 rich cells (lemma 2), the addition of the fourth plane cannot divide any of these rich cells into two new rich cells as this can occur at most zero times (this is the number of rich cells when three lines in a plane have a point in common), and $2^3 \leq 2^4 - 3 - 2$.

In the following we will use only $f(3,4)$ and $f(2,3)$. The general case can be found in [1]. $\square$

Corollary 1 $f(3,4) = 11$ *and* $f(4,5) = 26$.

3.1 Results for 2 dimensions

It has already been shown that $f(2,1) = 2$ and $f(2,2) = 4$ by lemma 2, and that $f(2,3) = 4$ by lemma 3.

Lemma 4 $f(2,4) = 2$ *and* $f(2,5) = 1$.

Proof

(i) This can be shown by case analysis.

(ii) Observe that any two convex sets in the plane can have at most four tangent lines in common. $\square$

Corollary 2 $f(2,k) = 1$ *whenever* $k \geq 5$

4 An Upper Bound

Theorem 5 *For* $n \geq d \geq 3$

$$f(d,n) \leq \frac{(n+8)^{d-2}}{(d-2)!}$$

Proof

The result holds for $d = 3$ by using the recurrence relation and the results for $f(3,4)$ and $d = 2$. It can be shown (using an inductive argument) that the result holds for $f(d,d)$ i.e. that for $d \geq 3$

$$f(d,d) = 2^d \leq \frac{(d+8)^{d-2}}{(d-2)!}$$

Inductively assume then that the result holds true in $(d-1)$ dimensions, and that in d dimensions the result holds for up to $(n-1)$ hyperplanes then

$$\begin{aligned} f(d,n) &\leq f(d,n-1)+f(d-1,n-1) \\ &\leq \frac{(n+7)^{d-2}}{(d-2)!}+\frac{(n+7)^{d-3}}{(d-3)!} \\ &\leq \frac{((n+7)+1)^{d-2}}{(d-2)!} \\ &= \frac{(n+8)^{d-2}}{(d-2)!} \end{aligned}$$

□

5 A Lower Bound

In this section we construct an example that gives a lower bound: $f(d,n) \geq \binom{n}{d-2}+\binom{n}{d-3}+\cdots+\binom{n}{0}$, where $n \geq d+1$ The method involves constructing an arrangement in space one dimension higher than that required and cutting this with a hyperplane to get an arrangement in $\mathbb{R}^d$. First consider an arrangement of $n \geq d+1$ hyperplanes through the origin in $\mathbb{R}^d$. Let $H_i = \{x :< a_i, x >= 0, a_i \in \mathbb{R}^d\}$. These planes dissect the space into cones $C = \{x \in \mathbb{R}^d :< \varepsilon_i a_i, x > \leq 0, i = 1,\ldots,n\}$, where $\varepsilon_1,\ldots,\varepsilon_n = +1$ or -1, is a sign sequence. Such a cone is rich if it has n facets. A simple duality argument shows that $\varepsilon_1,\ldots,\varepsilon_n$ determine a rich cone if and only if $\mathrm{Cone}\{\varepsilon_1 a_1,\ldots,\varepsilon_n a_n\}$ has n extreme rays, we denote this property by (*). Set $g(d,n)$ to be the maximum number of rich cones in a dissection by n planes in $\mathbb{R}^d$

If $n \geq d+1$ then

$$g(d,n) \geq 2\left\{\binom{n-1}{d-3}+\binom{n-1}{d-4}+\cdots+\binom{n-1}{0}\right\} := K(n,d)$$

Proof

Let $a_1,\ldots,a_n$, $n \geq d+1$, be lexicographically ordered on the moment curve. There are $K(n,d)$ sign sequences $\varepsilon_1,\ldots,\varepsilon_n$, $\varepsilon_i = \pm 1$, each with at most $(d-3)$ sign changes. We claim that all resulting sequences $\varepsilon_i a_i$ satisfy (*). This will prove claim 5 above. Assume that the sequence resulting from $\varepsilon_1, \ldots, \varepsilon_n$ does not satisfy the property (*). This implies that there is some j with the property $\varepsilon_j a_j \in \text{Cone}\{\varepsilon_i a_i : i \in \{1, \ldots, n\} - \{j\}\}$. Then by Carathéodory's Theorem [3] $\varepsilon_j a_j \in \text{Cone}\{\varepsilon_i a_i : i \in D\}$ where $D \subset \{1, \ldots, n\} - \{j\}$, and $|D| = d$. Set $D' = D \cup \{j\}$ and let $D' = \{i_1, \ldots i_{d+1}\}$ with $i_1 < i_2 < \cdots < i_{d+1}$. Thus:

$$0 = \sum_{k=1}^{d+1} \alpha_k \varepsilon_{i_k} a_{i_k} \text{with } \begin{array}{ll} \alpha_k \geq 0 & \text{if } i_k \neq j \\ \alpha_k = -1 & \text{if } i_k = j \end{array}$$

We know from the properties of the moment curve that in any such linear dependence the signs of the coefficients of the a_{i_k} alternate (and that none of them are zero) [5]. Let us call the intervals between sign changes in the ε_i sequence *blocks*. There are at most $d - 2$ blocks. If a block that does not contain j contains two (consecutive) i_k's, then the a_{i_k}'s have the same sign in the above linear dependence, which is impossible. If the block that contains j contains four (or more) i_k's then again two consecutive a_{i_k}'s have the same sign—again impossible. So if we count the number of a_{i_k}'s which can be distributed among the blocks we have: the number of a_{i_k}'s$\leq 3 + (d - 3) = d < d + 1$ Which is a contradiction. □

We use this result to show claim 5. Take the above example in $\mathbb{R}^{d+1}$ with $n + 1$ vectors $a_1, \ldots a_{n+1}$. In this arrangement the plane $<a_1, x>= 0$ supports a facet in each of (at least) $K(n + 1, d + 1)$ rich cones. So the plane $<a_1, x>= 1$ intersects (at least) half this number of the rich cones. Each such intersection is a rich cell in $\mathbb{R}^d = \{x :<a_1, x>= 1\}$, with the hyperplanes of the arrangement being $H_i \cap \mathbb{R}^d$, $i = 2, \ldots, n + 1$. □

This example provides a lower bound for $f(d, n)$ and finishes the proof of the theorem stated in the introduction.

6 Convex Position

We propose the following generalisations of the concept of convexity for k-dimensional flats in d-space.

Let $\mathcal{F}$ be a family of k-flats in $\mathbb{R}^d$ lying in general position. We say that $\mathcal{F}$ is in *convex position* if there is a compact convex body touching every member of $\mathcal{F}$ (see also [4]).

Obviously, any set of $n \leq d+1$ points in general position in $\mathbb{R}^d$ induces an $(n-1)$-dimensional simplex and is therefore in convex position. On the other hand, by Carathéodory's theorem, if all $(d+2)$-tuples of n distinct points in

d-space are in convex position, then all points are in convex position. Our original reason for studying rich cells was to establish some analagous results for k-flats in convex position, but we could only handle the planar case.

It is easy to see that any family of 4 lines in the plane is in convex position, and this is the largest number with this property.

Theorem 6 *If any 5 members of a finite family of lines in the plane are in convex position, then all of them are in convex position.*

Proof

Suppose, in order to obtain a contradiction, that there is a family $\mathcal{L} = \{l_1, \dots, l_{n+1}\}$ of $n+1$ lines, for some $n \geq 5$, which is not in convex position but any proper subfamily of $\mathcal{L}$ is in convex position.

Any family of $n \geq 5$ lines divides the plane into $\binom{n+1}{2} + 1$ cells. Observe that by lemma 4 at most one of these cells can contain a piece (i.e. a segment or a half-line) of each line on its boundary.

Suppose now that $n > 5$, and that l_{n+1} intersects the (unique) cell C determined by $\{l_1, \dots, l_n\}$, whose boundary contains a piece of each l_i, $1 \leq i \leq n$. Then l_{n+1} cuts C into two pieces, C_1 and C_2, and we can assume without loss of generality that C_1 has at least as many sides as C_2. Clearly, C_2 has at least one side (belonging to, say, l_n) which is not incident to the common boundary segment of C_1 and C_2. We can assume that C_2 has no other side with this property, otherwise deleting its supporting line from $\mathcal{L}$ we would obtain a subfamily in non-convex position.

Then if l_{n+1} meets C in a bounded line segment. Let l_1 and l_{n-1} denote the sides of C intersected by l_{n+1} (see Figure 2), and let C_1^1 and C_1^n be the uniquely determined cells containing a piece of every line in the arrangements $\mathcal{L} - \{l_1, l_n\}$ and $\mathcal{L} - \{l_{n-1}, l_n\}$, respectively. Obviously $C_1^1, C_1^n \supseteq C_1$ and at least one of them is not met by l_n. Thus, $\mathcal{L} - \{l_1\}$ or $\mathcal{L} - \{l_{n-1}\}$ is not in convex position, which is impossible.

Otherwise l_{n+1} meets C in a unbounded ray intersecting the side l_{n-1}, say. Let C_1^n be as before. If l_n does not meet C_1^n then $\mathcal{L} - \{l_{n-1}\}$ is not in convex position, contradiction. If on the other hand l_n meets C_1^n, then l_{n-1} meets the cell determined by $\mathcal{L} - \{l_{n-1}\}$ in a bounded line segment and a contradiction is obtained using the above.

Figure 1:

Suppose next that $n > 5$, and that l_{n+1} does not intersect C. Assume without loss of generality that the sides of C adjacent to its vertex closest to l_{n+1} belong to l_1 and l_n. Then $\mathcal{L} - \{l_2\}$ is in non-convex position, which is a contradiction.

The case $n = 5$ can be treated by case analysis. □

Remark 1

There is another way of showing that no family $\mathcal{L} = \{l_1, \ldots, l_{n+1}\}$ exists with the property required in the above proof, provided that n is sufficiently large. Assign to every l_i the unique cell C_i in the arrangement $\mathcal{L} - \{l_i\}$ whose boundary contains a piece of each line in $\mathcal{L} - \{l_i\}$. Let $C_i^* = C_i$ if l_i does not intersect C_i, otherwise let $C_i^* \subseteq C_i$ be a cell in the arrangement of $\mathcal{L}$ with at least $n/2 + 2$ sides. It is easy to show that at least $(n + 1)/5$ of the C_i^* are distinct (because each cell belongs to at most 5 indices). The total number of sides of the C_i^* is at least $\frac{1}{5}(n + 1)(\frac{n}{2} + 2)$.

On the other hand it is well known that $n + 1$ cells in an arrangement of $n + 1$ lines cannot have more than $O(n^{4/3})$ sides [8] and [2], a contradiction if n is large enough.

The first part of this argument generalises to higher dimensions, but the total number of facets of $c_1 n$ cells in an arrangement of $n + 1$ hyperplanes in $\mathbb{R}^d$, $(d \geq 3)$ can be as large as $c_2 n^2$.

If we want to generalise the (first) proof of Theorem 6 to families of hyperplanes in $\mathbb{R}^d$ $(d \geq 3)$, then the problem is that by the example given in section 5 there can be more than one cell in an arrangement $\mathcal{H}$ of hyperplanes whose boundary contains a portion of every member of $\mathcal{H}$.

Remark 2

Instead of considering convex position, it may be more convenient to study projectively convex position of hyperplanes, that is if there exists a permissable projective transformation that maps the family of hyperplanes onto a family in convex position. In this formulation the problem is closely related to a question of McMullen [6], [7]. In fact the results in these papers provide a lower bound for the possible Carathéodory number.

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