The maximum number of times the same distance can occur among the vertices of a convex n-gon is $O(n \log n)$

Peter Braß[1] *and János Pach*[2]

[1]FU Berlin
[2] Courant Institute, NYU

Abstract

We present a short proof of Füredi's theorem [F] stated in the title.

Proof

Denote by $f(n)$ the maximum number of times the unit distance can occur among n points in convex position in the plane. Let $p_1, p_2, \ldots, p_n$, in this cyclic order, be the vertices of a convex polygon, for which the maximum is attained. Let G denote the geometric graph obtained by connecting two points of P by a straight-line segment (*edge*) if and only if their distance is *one.* Pick a point p_i *antipodal* to p_1, i.e., assume that there are two parallel lines, ℓ and ℓ', passing through p_1 and p_i, resp., such that all elements of P are in the strip between them.

We claim that all but at most $2n$ edges of G cross p_1p_i. To verify this, suppose without loss of generality that ℓ and ℓ' are parallel to the x-axis, and that no edge of G is parallel to the y-axis. Color any edge of G *red* if its slope is positive and *blue* otherwise. Assign every red edge lying in the closed half-plane to the left (right) of p_1p_i to its left (right) endpoint. It is easy to see that every element of P is assigned to at most one red edge. Therefore, the number of red edges not crossing p_1p_i is at most n. The same is true for the blue edges, which proves the claim.

We can assume without loss of generality that $i > n/2$, otherwise the numbering of the vertices can be reversed. Take a point p_j antipodal to $p_{\lceil i/2 \rceil}$. As above, there

[2]Supported by NSF grant CR-97-32101, PSC-CUNY Research Award 61392-0030, and OTKA-T-020914.

are at most $2n$ edges of G, which do not cross $p_{\lceil i/2 \rceil} p_j$. Every edge of G, crossing both $p_1 p_i$ and $p_{\lceil i/2 \rceil}$, connects a pair of points in

$$P_1 := \{p_2, p_3, \ldots, p_{\lceil i/2 \rceil - 1}\} \cup \{p_{i+1}, p_{i+2}, \ldots, p_{j-1}\}$$

or in

$$P_2 := \{p_{\lceil i/2 \rceil + 1}, p_{\lceil i/2 \rceil + 2}, \ldots, p_{i-1}\} \cup \{p_{j+1}, p_{j+2}, \ldots, p_n\}.$$

Thus, we have

$$f(n) = |E(G)| \leq f(|P_1|) + f(|P_2|) + 4n.$$

Using the facts that $|P_1| + |P_2| = n - 4$ and $\min(|P_1|, |P_2|) \geq \frac{n-7}{4}$, the theorem follows by induction. $\square$

It is an exciting open problem to decide whether $f(n) = O(n)$ holds. The best known general lower bound, $f(n) \geq 2n - 7$, is due to Edelsbrunner and Hajnal [EH].

References

[EH] H. Edelsbrunner and P. Hajnal: A lower bound on the number of unit distances between the points of a convex polygon, *J. Combinatorial Theory, Series A* **56**, 312–316.

[F] Z. Füredi: The maximum number of unit distance in a convex n-gon, *J. Combinatorial Theory, Series A* **55** (1990), 316–320.