Comment on Fox News

János Pach[*] and **Géza Tóth**[†]
Rényi Institute, Hungarian Academy of Sciences

Abstract

Does there exist a constant $c > 0$ such that any family of n continuous arcs in the plane, any pair of which intersect at most once, has two disjoint subfamilies A and B with $|A|, |B| \geq cn$ with the property that either every element of A intersects all elements of B or no element of A intersects any element of B? Based on a recent result of Fox, we show that the answer is no if we drop the condition that two arcs can cross at most once.

1 Introduction

It was shown in [4] that any family of n segments in the plane has two disjoint subfamilies A and B, each of size at least constant times n, such that either every element of A intersects all elements of B or no element of A intersects any element of B. In [1], this result was extended to families of algebraic curves with bounded degree at most D, where the corresponding constant depends on D.

More generally, let G be the intersection graph of n d-dimensional semialgebraic sets of degree at most D. Then there exist two disjoint subsets $A, B \subset V(G)$ such that $|A|, |B| \geq c(d, D)n$ and one of the following two conditions is satisfied:

1. $ab \in E(G)$ for all $a \in A, b \in B$,

[*]Supported by NSF grant CCR-0514079 and grants from NSA, PSC-CUNY, Hungarian Research Foundation, and BSF.
[†]Supported by OTKA-T-038397 and OTKA-T-046246.

2. $ab \notin E(G)$ for all $a \in A, b \in B$.

Here $c(d, D)$ is a positive constant depending only on d and D.

It is not completely clear whether the assumption that the sets are semialgebraic can be weakened. For example, a similar result may hold for intersection graphs of plane convex sets. Clearly, the same theorem is false for intersection graphs of three-dimensional convex bodies, because *any* finite graph can be represented in such a way, and a random graph G with n vertices almost surely does not have $A, B \subset V(G)$ satisfying conditions 1 or 2 with $|A|, |B| \geq c \log n$, if c is large enough.

It would be interesting to analyze intersection graphs of continuous arcs in the plane. (These are often called "string graphs" in the literature [2].) We have been unable to answer the following question even for $k = 1$, that is, for pseudo-segments.

Problem 1.1. *Is it true that any family of n continuous arcs in the plane, any pair of which intersect at most k times, has two disjoint subfamilies A and B with $|A|, |B| \geq c_k n$ such that either every element of A intersects all elements of B or no element of A intersects any element of B? (Here $c_k > 0$ is a suitable constant.)*

It follows from a beautiful recent result of Jacob Fox [3] (see Theorem 2.2 below) that the answer to the above question is negative if we drop the condition on pairwise intersections.

Proposition 1.2. *Fix $\varepsilon \in (0, 1)$. For every n, there is a family of n continuous real functions defined on $[0, 1]$ such that their intersection graph G has no complete bipartite subgraph with at least $c(\varepsilon)\frac{n}{\log n}$ vertices in each of its vertex classes, and every vertex of G is connected to all but at most n^{ε} other vertices.*

Obviously, the last condition implies that G has no two disjoint nonempty sets of vertices A and B with $|A \cup B| > n^{\varepsilon}$ such that no vertex in A is connected to any element of B by an edge.

2 Proof of Proposition 1.2

We need a simple representation lemma.

Lemma 2.1. *The elements of every finite partially ordered set* $(\{p_1, p_2, \ldots\}, <)$ *can be represented by continuous real functions* $f_1, f_2, \ldots$ *defined on the interval* $[0,1]$ *such that* $f_i(x) < f_j(x)$ *for every* x *if and only if* $p_i < p_j \ \ (i \neq j)$.

Moreover, we can assume that the graphs of any pair of functions f_i *and* f_j *are either disjoint or have finitely many points in common, at which they properly cross.*

Proof. Let $P = \{p_1, p_2, \ldots p_\ell\}$. We describe a recursive construction with the additional property that for any extension of $(P, <)$ to a total order $p_{k(1)} < p_{k(2)} < \ldots < p_{k(\ell)}$, there exists $x \in [0,1]$ such that $f_{k(1)}(x) < f_{k(2)}(x) < \ldots < f_{k(\ell)}(x)$.

The proof is by induction on the number of elements of P. For $\ell = 1$, there is nothing to prove. For $\ell = 2$, there are two possibilities. If $p_1 < p_2$, then the functions $f_1 \equiv 1$, $f_2 \equiv 2$ meet the requirements. If p_1 and p_2 are incomparable, then let $f_1(x) = x$, $f_2(x) = 1 - x$. Now $(P, <)$ can be extended to a total order in two different ways. Accordingly, $f_1(x) < f_2(x)$ at $x = 0$ and $f_2(x) < f_1(x)$ at $x = 1$.

Let $\ell \geq 3$, and suppose without loss of generality that p_ℓ is a minimal element of P. Assume recursively that we have already constructed continuous real functions $f_1, f_2, \ldots, f_{\ell-1}$ with the required properties representing the elements of the partially ordered set $(P \setminus \{p_\ell\}, <)$. Consider now an extension of $(P, <)$ to a total order $p_{k(1)} < p_{k(2)} < \ldots < p_{k(\ell)}$. Clearly, p_ℓ appears in this sequence, i.e., $\ell = k(m)$ for some $1 \leq m \leq \ell$. By our assumption, there exists $x \in [0,1]$ such that

$$f_{k(1)}(x) < \ldots < f_{k(m-1)}(x) < f_{k(m+1)}(x) < \ldots < f_{k(\ell)}.$$

In fact, there exists a whole interval $I \subset [0,1]$ such that the above inequalities hold for all $x \in I$. Now pick a point $x^* \in I$ and a number y^* such that $f_{k(m-1)}(x^*) < y^* < f_{k(m+1)}(x^*)$, and define

$$f_\ell(x^*) := y^*.$$

Repeating this procedure for every permutation $(k(1), k(2), \ldots, k(\ell))$ for which $p_{k(1)} < p_{k(2)} < \ldots < p_{k(\ell)}$ is an extension of $(P, <)$ to a total order, we define the function f_ℓ at finitely many points. (To avoid inconsistencies, we can make sure that we pick a different point x^* for each permutation.)

It remains to verify that this partially defined function can be extended to a continuous function $f_\ell : [0,1] \to \mathbf{R}$ meeting the requirements. The following two conditions must be satisfied:

1. if $p_\ell < p_j$ in $(P,<)$ for some $j \neq \ell$, then $f_\ell(x) < f_j(x)$ for all $x \in [0,1]$;
2. if p_ℓ and p_j are incomparable in $(P,<)$ for some $j \neq \ell$, then the graphs of f_ℓ and f_j cross each other.

Notice that each point (x^*, y^*) constructed during the above procedure lies below the lower envelope (pointwise minimum) of the functions $f_j(x)$ over all j for which $p_j > p_\ell$ in $(P,<)$. Pick a point $x_0 \in [0,1]$ distinct from all previously selected points $x^* \in [0,1]$, and let $f_\ell(x_0) := y_0$ for some

$$y_0 < \min_{1 \leq j < \ell} f_j(x_0).$$

Extend f_ℓ to a continuous function on $[0,1]$ whose graph lies strictly below

$$\min\{f_j(x) \ : \ \text{for all } j \text{ such that } p_j > p_\ell\}.$$

Obviously, f_ℓ satisfies condition 1. To see that condition 2 is also satisfied, fix an index j such that p_ℓ and p_j are incomparable in $(P,<)$. Consider an extension of $(P,<)$ to a total order in which $p_j < p_\ell$. It follows from our construction that there exists a point $x \in [0,1]$ at which the values $f_i(x)$ are in the same total order as the elements p_i $(1 \leq i \leq \ell)$. In particular, we have $f_j(x) < f_\ell(x)$. On the other hand, by definition, $f_\ell(x_0) = y_0 < f_j(x_0)$. Therefore, the graphs of f_ℓ and f_j must cross each other, completing the proof. $\Box$

Theorem 2.2. (Fox) *Fix $\varepsilon \in (0,1)$. For every n, there is a partially ordered set $(P,<)$ of size n with the following two properties. (i) There are no two disjoint subsets $A, B \subset P$ such that $|A|, |B| \geq c(\varepsilon)\frac{n}{\log n}$ and no element of A is comparable to any element of B. (ii) Every element of P is comparable to at most n^ε other elements.* $\Box$

To deduce Proposition 1.2, apply Lemma 2.1 to the partially ordered set whose existence is guaranteed by Theorem 2.2. To see that the intersection graph G of the resulting functions meets the requirements, it is enough to notice that two vertices of G are connected by an edge if and only if the corresponding elements of P are incomparable.

References

[1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić, and M. Sharir: Crossing patterns of semi-algebraic sets, *Journal of Combinatorial Theory Ser. A*, accepted.

[2] P. Brass, W. Moser, and J. Pach: *Research Problems in Discrete Geometry*, Springer-Verlag, New York, 200.

[3] J. Fox: A bipartite analogue of Dilworth's theorem, manuscript, 2005.

[4] J. Pach and J. Solymosi: Crossing patterns of segments, *Journal of Combinatorial Theory Ser. A* **96** (2001), 316–325.