Folding and turning along geodesics in a convex surface

*János Pach**
Courant Institute, New York University,
251 Mercer Street New York, NY 10012

Let γ be a *shortest path* connecting two points, p and p', on the surface of a three-dimensional convex polytope P. Then γ is a polygonal path $p_0p_1p_2\dots p_m$, where $p_0 = p, p_m = p'$ and each internal vertex p_i $(0 < i < m)$ belongs to an edge of P denoted by e_i. For every i $(0 < i < m)$, let $\pi - \varphi_i$ denote the dihedral angle between the faces of P meeting at e_i, and let $\pi - \tau_i$ stand for $\angle p_{i-1}p_ip_{i+1}$. φ_i and τ_i are called the *folding angle* and the *turning angle* of γ at p_i, respectively. Accordingly, define the *total folding angle* of γ and the *total turning angle* of γ as

$$\varphi(\gamma) = \sum_{0<i<m} \varphi_i, \quad \tau(\gamma) = \sum_{0<i<m} \tau_i.$$

For every i, we have

$$0 < \tau_i \leq \varphi_i < \pi.$$

Thus, the total turning angle of γ cannot exceed the total folding angle of γ. (See Figure 1.)

Recently, Sariel Har-Peled and Micha Sharir have raised the following interesting problem.

Problem. [AHSV96] Does there exist an absolute constant K such that the total turning angle of every shortest path γ on the surface of any three-dimensional convex polytope P is at most K?

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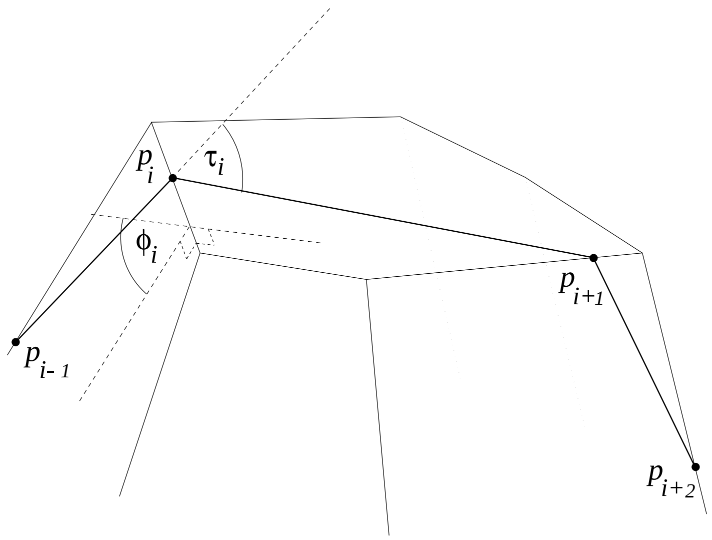

Fig. 1.

The stronger question, whether there is a finite upper bound on the total *folding* angle of every shortest path on every convex polyhedral surface, was asked in A. V. Pogorelov's famous book [P69].

We show that the answer to this latter question is in the negative.

Theorem. *For every K, there exist a three-dimensional convex polytope P and a shortest path on the surface of P, whose total folding angle is at least K.*

Proof. Fix an integer $n > 2K$. Let e_1, e_2, e_3 be three points (vectors) forming an equilateral triangle in the (x, y)–plane:

$$e_1 = (1, 0), \quad e_2 = (-1/2, \sqrt{3}/2), \quad e_3 = (-1/2, -\sqrt{3}/2).$$

For every $0 \le i \le n$ and for every $1 \le j \le 3$, let

$$v_{ij} = (-1/3)^i e_j,$$

and let q_{ij} denote the projection of v_{ij} onto the convex surface

$$z = \varepsilon \left((x^2 + y^2)^{3/2} - 1/2 \right),$$

parallel to the z–axis. Explicitly,

$$\begin{aligned}
q_{i1} &= \left((-1/3)^i, 0, \varepsilon((1/3)^{3i} - 1/2)\right),\\
q_{i2} &= \left(-(-1/3)^i/2, (-1/3)^i\sqrt{3}/2, \varepsilon((1/3)^{3i} - 1/2)\right),\\
q_{i3} &= \left(-(-1/3)^i/2, -(-1/3)^i\sqrt{3}/2, \varepsilon((1/3)^{3i} - 1/2)\right),
\end{aligned}$$

where ε is a small positive constant to be specified later.

Let Q be the convex hull of all points $q_{ij}, 0 \leq i \leq n, 1 \leq j \leq 3$. Clearly, every q_{ij} is a vertex of the polytope Q, and Q contains the origin $(0,0,0)$ in its interior. It is easy to verify that two distinct vertices, q_{ij} and q_{kl}, are connected by an edge of Q if and only if

(1) $i = k$, or

(2) $|i - k| = 1$ and $j \neq l$.

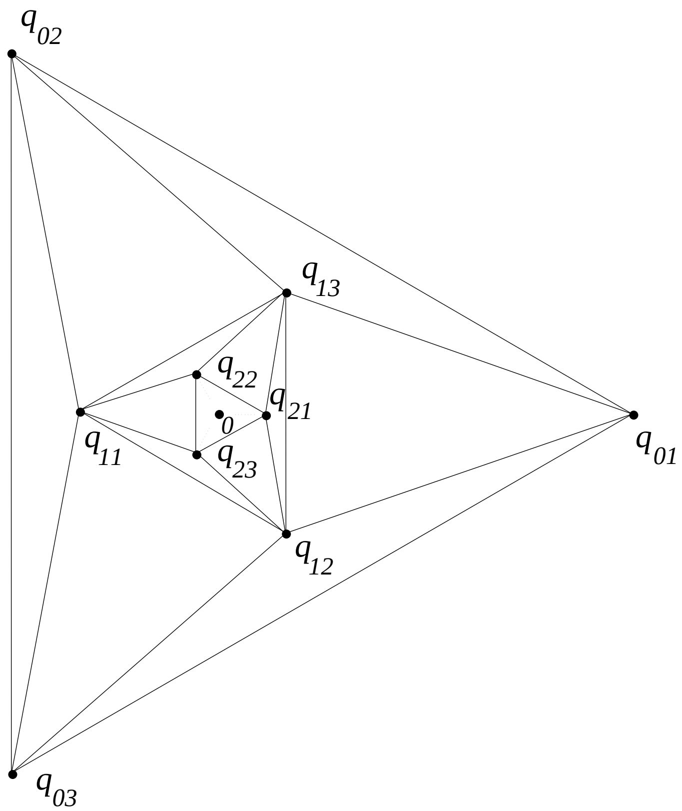

Fig. 2.

The orthogonal projection of the skeleton (edge structure) of Q onto the (x, y)–plane is depicted on Figure 2. In fact, if ε is very small, then Q hardly differs from its projection.

Let Q^* denote the *polar* polytope of Q, i.e., let

$$Q^* = \{p \in \Re^3 \mid \langle p, q\rangle \leq 1 \text{ for every } q \in Q\}.$$

It is well known [G67], [MS71] that there is a one-to-one correspondence between the vertices of Q and the faces of Q^* such that

(i) two vertices of Q are joined by an edge if and only if the corresponding two faces of Q^* are adjacent;

(ii) the vector representing any vertex of Q is perpendicular to the corresponding face of Q^*.

Thus, the angle between any two vectors representing adjacent vertices of Q is equal to the *folding angle* (i.e., π minus the dihedral angle) between the corresponding two faces of Q^*. It follows from the definition of Q that this angle can be bounded from below by any number smaller than $\pi/3$, provided that ε is sufficiently small. In particular, we can fix $\varepsilon > 0$ so that every edge of Q can be seen from the origin at an angle larger than $\pi/4$. Consequently, the folding angle between any two adjacent faces of Q^* is larger than $\pi/4$.

Let p and p' be internal points of the faces of Q^* corresponding to q_{01} and q_{n1}, respectively. Observe that any path connecting q_{01} and q_{n1} in the skeleton of Q consists of at least n edges. Therefore, any (shortest) path γ connecting p and p' on the surface of Q^* crosses at least n edges of Q^*. Thus, the total folding angle of any such path is larger than $n\pi/4 > 2K\pi/4 > K$, showing that $P = Q^*$ meets the requirements of the Theorem. $\Box$

The problem of Har-Peled and Sharir, mentioned earlier, is still open.

References

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