Cellular telephone networks and random maps in hypergraphs

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Abstract

Let $\mathcal{H} = (V, E)$ be an r-uniform hypergraph of size n such that each edge of $\mathcal{H}$ meets at most d others. A finite map $f : X \to E$ induces a bipartite graph $G_{\mathcal{H},f} = (V_f, E_f)$ with vertex set $V_f = X \cup Y$ where $Y = \cup E$, and with edge set $E_f = \{\{x, y\} : x \in X, y \in f(x)\}$. We study matchings in the bipartite graph induced by a random f. The study was suggested by consideration of the call sequence acceptance behavior of a load sharing system for cellular telephone networks, invented by Matula and Yang.

KEYWORDS cellular telephone, matching, 0–1 law, limit probability distribution, hypergeometric function

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1 Introduction

Matula and Yang [4] designed a load sharing algorithm which enables a cellular telephone network to dynamically reconnect a sequence of telephone calls to neighboring transceivers to free a transceiver for a new call entering the network. They described some possible implementations of their system, in which a number of cellular telephone transceivers are deployed in triangular and hexagonal planar arrangements with multiple cellular transceiver sites per cell.

In subsection 1.1, we describe a hypergraph model of Matula and Yang's system. In Section 2, we summarize our results. Subsection 2.1 contains some simple probabilistic statements on the asymptotic number of failed attempts in a random sequence of calls in the hypergraph model. In Subsection 2.2 we give exact enumeration formulae in the one-dimensional case, some of which were obtained with the help of a computer. The proofs are presented in Section 3.

1.1 Hypergraph model

Let $\mathcal{H} = (V, E)$ be a hypergraph with vertex set V and edge set E. If every edge of $\mathcal{H}$ consists of precisely r vertices, then $\mathcal{H}$ is called an r-*uniform* hypergraph.

The hypergraph model of the system of Matula and Yang is a connected r-uniform hypergraph $\mathcal{H} = (V, E)$ with $|E| = n$ (hyper)edges such that every edge intersects at most d others. Each vertex of $\mathcal{H}$ represents a cellular telephone transceiver tower capable of receiving a *single* call. Each edge of $\mathcal{H}$ represents a collection of towers that service a region. In the sequel, we will often refer to a hypergraph edge as a *cell* to distinguish it from a graph edge and to emphasize the analogy with the polygonal model.

Let m be a positive integer, and let $[m] = \{1, \ldots, m\}$. A sequence of calls to various cells of the system is represented by a finite function $f : F \to E$, where $F \subseteq [m]$ for some positive integer m. The elements of $[m]$ are thought of as calls. Each call $j \in F$ is assigned to the cell $f(j) = e$. The number of calls assigned to $e \in E$ is the number of times that f takes the value e. Given $\mathcal{H}$ and f, a bipartite graph $\Gamma_f = (V_f, E_f)$ can be associated with f in a natural way, as follows. Define the vertex set of Γ_f by $V_f = F \cup V$, and its edge set by $E_f = \{\{j, v\} : j \in F \text{ and } v \in f(j) \subseteq V\}$. A function $f : F \to E$ is *admissible* iff there exists a matching from F to V in Γ_f. It follows immediately from the König-Hall Lemma (Marriage Theorem) that f is admissible iff for every subset A of F, $|A| \leq |\Gamma_f(A)|$, where $\Gamma_f(A)$ denotes the set of vertices in V adjacent in Γ_f to some vertex of A. Given a finite map $f : F \to E$, we define the *admissible domain* A_f of f to be the lexicographically smallest subset A of F of maximum cardinality such that the restriction $f|A$ of f to A is admissible. A function $f : F \to E$ *rejects* t iff $t \in F - A_f$. The *number of rejections of* f is defined to be $|F| - |A_f|$.

Let $f : F \to E$ be a map, $F \subseteq [m]$. Given any $1 \leq j \leq m$, $e \in E$, we say that f *saturates* e *at* j iff $j \in A_f$ and for any map $g : G \to E$ with $g|A_f = f|A_f$, where G is the union of A_f with a non-empty finite set of positive integers strictly greater than the maximum element of A_f, and for any $k \in G$ such that $k > j$, $g(k) = e$ implies that g rejects k.

Matula and Yang's patent [4] illustrates a possible implementation of their call switching invention in a cellular telephone system with hexagonal cells. Each hexagonal cell contains seven transceiver sites, one centrally located, and one at each vertex. The collection of cells and transceiver sites in this example form a 7-regular hypergraph in which each edge meets at most 6 other edges.

Suppose that $\mathcal{H}$ is a hypergraph as above. We wish to compute the probability that a random uniformly distributed sequence of m calls arranged in the n edges of $\mathcal{H}$ will be admissible. We also wish to compute a limit probability distribution of the admissible functions when m and n are large.

2 Results

2.1 Probabilistic approach

Using the probabilistic method [1], we investigate the asymptotic behavior of the expected number of rejected calls for a random uniformly distributed function $f : [m] \to E$, where E is the edge set of an r-uniform hypergraph, as m and $|E| = n$ tend to infinity. The theorems in this section pertain only to r–uniform hypergraphs $\mathcal{H} = (V, E)$ satisfying the following condition.

Condition 1 *For every non-empty subset $A \subseteq E$ of edges of $\mathcal{H}$, $|A| + r - 1 \leq |\cup A|$.*

According to the remark at the end of Subsection 1.1, most of the natural instances of the polygonal model meet this requirement.

We use the notation $f \ll g$, $f \gg g$, and $f \sim g$ to signify that $\lim_{n\to\infty} f(n)/g(n)$ is 0, ∞, and 1, respectively, where f and g are real valued functions defined for large real or integral arguments.

Theorem 2 *Let $\mathcal{H} = (V, E)$ be an r–uniform hypergraph with $|E| = n$ cells such that every cell intersects at most d others, and assume that $\mathcal{H}$ satisfies Condition 1. In the discrete probability space of all maps $f : [m] \to E$ with the uniform distribution, let R_k denote the event that the k-th call is rejected, let X_k be the indicator random variable for R_k, and define $\overline{X}_k = \sum_{j=1}^{k} X_j$. Let d and r be fixed, and let n and $m = m(n)$ be positive integers tending to infinity.*

Then

$$\lim_{n\to\infty} E[\overline{X}_m] = \begin{cases} 0 & \text{if } m \ll n^{\frac{r}{r+1}}, \\ \infty & \text{if } m \gg n^{\frac{r}{r+1}}. \end{cases}$$

Taking into account that $\overline{X}_m$ is a non-negative integer valued random variable, we have that $\Pr[\overline{X}_m > 0] \leq E[\overline{X}_m]$, i.e., we obtain the following

Corollary 3 *$\overline{X}_m = 0$ almost surely if $m \ll n^{\frac{r}{r+1}}$;*

We have been unable to show that if $m \gg n^{\frac{r}{r+1}}$, then $\overline{X}_m \sim E[\overline{X}_m]$, i.e., for every $\epsilon > 0$, we have

$$\lim_{m\to\infty} \Pr[|\overline{X}_m / E[\overline{X}_m] - 1| \geq \epsilon] = 0.$$

However, applying martingale techniques, we can show

Theorem 4 *$\overline{X}_m \sim E[\overline{X}_m]$ almost surely if $m \gg n^{\frac{r}{r+1/2}}$.*

2.2 Algebraic approach

We consider the simplest cellular telephone system, modeled by a connected 2-uniform hypergraph $\mathcal{H}$ of maximum degree 2. There are two possibilities for such hypergraphs: paths and cycles. We formulate our asymptotic result for paths, but it also applies to cycles.

For a path of n edges, a sequence will fail to be admissible if it assigns some edge three or more calls, or if it assigns each edge at most two calls, and there is a connected sequence of at least two edges starting and ending with an edge containing two calls, and such that every other edge in the sequence is assigned precisely one call. An admissible sequence of $m \geq 1$ calls is an "almost injective" function; that is, a function $f : [m] \to [n]$ for which there exist $g : [m] \to \{0,1\}$ and $h : [m] \to [n+1]$ with h injective such that $f + g = h$.

The main result of this section is a limit probability distribution for the admissible functions. Given m, n positive, consider the discrete probability space of maps $f : [m] \to [n]$, with the uniform probability distribution.

Theorem 5 *Let A be a positive constant. Then*

$$\lim_{n\to\infty} \Pr[f : [m] \to [n] \text{ is admissible }] = \begin{cases} 1 & \text{if } m \ll n^{2/3}, \\ e^{-A^3/6} & \text{if } m \sim An^{2/3}, \\ 0 & \text{if } m \gg n^{2/3}. \end{cases}$$

To state the enumeration results used for the proof of Theorem 5, we need some notation for hypergeometric functions. The *shifted factorial*, also called the *Pochhammer symbol*, is denoted by $(a)_k$, and defined for $a \in \mathbb{R}$ by $(a)_0 = 1$, and by $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$. The generalized hypergeometric function with p numerator parameters $a_1, \ldots, a_p$, and q denominator parameters $b_1, \ldots, b_q$ is denoted and defined by

$${}_pF_q\left[\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}.$$

Proposition 6 *Let $A(m,n)$ denote the number of admissible functions $f : [m] \to [n]$, where $1 \leq m \leq n+1$. Then $A(m,n) = M(m, n-m) + N(m,n)$, where*

$$M(m,z) = (z+1)_m + m! \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2^k} \sum_{s=0}^{m-2k} \binom{s+k+z}{s \quad k \quad z} \binom{m-s-k-1}{k-1}$$

and where

$$N(m,n) = \frac{1}{2} \sum_{k=1}^{m-1} M(m-k-1, n-m+1)(m-k)_{k+1}.$$

By applying Petkovsek's Mathematica implementation of Gosper's algorithm [5, 6] for summing hypergeometric series, we arrive at the following hypergeometric identity for the number of admissible functions.

Proposition 7 *Let $1 \leq m \leq n+1$. Then,*

$$A(m,n) = (n-m+1)_m \cdot {}_3F_2 \left[\begin{array}{c} n-m+1, -m/2, -(m-1)/2 \\ (n-m+1)/2, (n-m+2)/2 \end{array} ; \frac{1}{2} \right]$$
$$+\frac{m!}{2} \sum_{k=1}^{m-1} \binom{n-k}{n-m+1} {}_3F_2 \left[\begin{array}{c} n-m+2, (k-m+1)/2, (k-m+2)/2 \\ (n-m+2)/2, (n-m+3)/2 \end{array} ; \frac{1}{2} \right].$$

Richard Stanley [7] has shown that the number $A(m,n)$ of admissible functions $f:[m] \to [n]$ for $1 \leq m \leq m+1$ satisfies

$$\sum_{m,n=1}^{\infty} \frac{A(m,n)}{m!} x^m y^n = \frac{1 - xy + (x^2 y)/2}{1 - y - 2xy + xy^2 + (x^2y^2)/2}. \quad (1)$$

Herbert Wilf [9] pointed out that Stanley's rational generating function (1) implies that $A(m,n)$ satisfies the recurrence

$$\begin{aligned} A(m.n) &= A(m,n-1) + 2mA(m-1,n-1) \\ &\quad - mA(m-1,n-2) - \binom{m}{2} A(m-2,n-2) \end{aligned}$$

for $m,n \geq 3$.

3 Proofs

3.1 Proof of Theorem 2

Let $\mathcal{H} = (V,E)$ be a connected r-uniform hypergraph of n cells (hyperedges) such that every cell intersects at most d others, and $\mathcal{H}$ satisfies Condition 1 in Subsection 2.1.

Form the discrete probability space of all maps $f:[m] \to E$, with each map equally likely. Define the random variable Y_j to be the number of previously not saturated cells that become saturated precisely at the j-th call. Also, let $\overline{Y}_k = \sum_{j=1}^{k} Y_j$. Note if the j-th call is rejected, then $Y_j = 0$. We write $f = \Theta(g)$ provided $f = O(g)$ and $g = O(f)$.

A connected set U of u cells of E will be called a *u–cluster*. A pair (U,e) consisting of a u–cluster U together with a choice of an edge $e \in U$ is called a *pointed u–cluster*. The distinguished edge e is called the *point* of (U,e).

Lemma 8 *For any $1 \leq u \leq n$, the number of u–clusters in $\mathcal{H}$ is bounded above by*

$$\frac{n}{u} \cdot \frac{(3d)^u}{(d-1)u+1}.$$

Proof. Fix a cell $e \in E$. The number of pointed u–clusters whose point is e is bounded above by the number of rooted d–ary trees with u nodes, which is known [3] to be

$$\binom{du}{u}\frac{1}{(d-1)u+1} \leq \left(\frac{3du}{u}\right)^u \frac{1}{(d-1)u+1}.$$

■

Note that for $1 \leq j < r$, $Y_j = 0$, since no cell can be saturated by fewer than r calls.

Lemma 9 *Let r and d be fixed. For $r \leq j \leq m \ll n$,*

$$E[Y_j] = \Theta((j/n)^{r-1}).$$

Proof. Let $U \subseteq E$ and let (U, e) be a pointed u-cluster. Let $S_{j,(U,e)}$ be the set of all maps $f : [m] \to E$ whose j-th call is in e, meaning that $f(j) = e$, and such that f saturates each of the $u = |U|$ cells of U at j and no other cells of E. Define

$$S_{j,U} = \bigcup_{e \in U} S_{j,(U,e)}.$$

$S_{j,U}$ is the set of maps $f : [m] \to E$ that saturate the u-cluster U at j. By definition,

$$E[Y_j] = \sum_{\substack{1 \leq u \leq j \\ |U|=u}} u \Pr[S_{j,U}] = \sum_{\substack{1 \leq u \leq j \\ |U|=u}} u \sum_{e \in U} \Pr[S_{j,(U,e)}]$$

where the first sum is over all u-clusters U for u from 1 to j.

Let (U, e) be a pointed u-cluster. We overestimate the number of sequences f of length m that saturate U when the j-th call of f arrives. Let $v(U)$ be the number of vertices of U. There will be $v(U)$ calls connected to the $v(U)$ vertices of U when the j-th call of f arrives. Order the $v(U)$ vertices of U so that the last vertex is one of the r vertices of e. There are $(v(U) - 1)!r$ such orderings. Each of the initial $v(U) - 1$ calls must appear in one of the at most d cells containing the vertex to which it is assigned. The remaining calls prior to j are assigned to cells in $E - U$, and the $m - j$ calls after the j–th can be assigned to any of the n cells of E.

Consequently,

$$|S_{j,(U,e)}| \leq \{(v(U) - 1)!r\} \binom{j-1}{v(U)-1} d^{v(U)-1} n^{j-v(U)} n^{m-j}. \tag{2}$$

Recalling that d and r are fixed and that $1 \leq j \leq m \ll n$, we make the following estimates, noting that we can always take $n > dj$.

$$E[Y_j] = \sum_{1\le u\le j} u \sum_{|U|=u} \frac{|S_{j,U}|}{n^m} \le \sum_{1\le u\le j} u \sum_{|U|=u} \sum_{e\in U} \frac{r}{n}\left(\frac{d}{n}\right)^{v(U)-1} \frac{(j-1)!}{(j-v(U))!} \quad (3)$$

$$= O\left(\sum_{1\le u\le j} \frac{u^2}{n}\left(\frac{dj}{n}\right)^{u-1} \sum_{|U|=u} \left(\frac{dj}{n}\right)^{v(U)-u}\right)$$

$$= O\left(\sum_{1\le u\le j} \frac{u^2}{n}\left(\frac{dj}{n}\right)^{u-1} \sum_{|U|=u} \left(\frac{dj}{n}\right)^{r-1}\right) \quad (4)$$

$$= O\left(d^{r-1}\left(\frac{j}{n}\right)^{r-1} \sum_{1\le u\le j} \frac{u^2}{n}\left(\frac{dj}{n}\right)^{u-1} \sum_{|U|=u} 1\right)$$

$$= O\left(\left(\frac{j}{n}\right)^{r-1} \sum_{1\le u\le j} \frac{u^2}{n}\left(\frac{dj}{n}\right)^{u-1} \frac{n}{u}\frac{(3d)^u}{(d-1)u+1}\right) \quad (5)$$

$$= O\left(\left(\frac{j}{n}\right)^{r-1} \sum_{u=0}^{\infty} \left(\frac{3d^2 j}{n}\right)^{u}\right) = O\left(\frac{j}{n}\right)^{r-1}, \quad (6)$$

where (3) follows from (2); (4) follows from Condition 1, which in the current notation is $r-1 \le |\cup U| - |U| = v(U) - u$ for a u–cluster U; and (5) follows from the preceding lemma. (6) follows by choosing δ with $0 < \delta < 1$ and n so large that $3d^2 j/n < \delta$. Then the geometric series in (6) converges and approaches 1 as $\delta \to 0$.

To obtain a lower bound for $E[Y_j]$, we underestimate the probability that a function $f : [m] \to E$ saturates some 1-cluster at the j-th call. Let $j \le m \le n$, and let $S_{j,e}$ be the set of maps which saturate the 1-cluster consisting of the cell e at j. If $f \in S_{j,e}$, then necessarily $r \le j$, and there must be a subset S of $[j-1]$ of size $r-1$ such that f sends S to the at most d cells adjacent to $f(j) = e$. We underestimate $S_{j,e}$ by counting maps $f : [m] \to E$ that send all of the calls of $S \cup \{j\}$ to e, and which send the remaining calls to cells disjoint from e. This is possible because there are at least $n-j-1$ cells disjoint from e. Since $\mathcal{H}$ has n cells,

$$\frac{n(n-d-1)^{j-r}}{n^j}\binom{j-1}{r-1} \le \Pr(S_{j,1}) \le E[Y_j].$$

By Stirling's formula, for some constant $C > 0$,

$$C\left(\frac{n-d-1}{n}\right)^{j-1}\left(\frac{j-1}{n-d-1}\right)^{r-1} \le E[Y_j]. \quad (7)$$

For $j \ll n$, $((n-d-1)/n)^{j-1}$ converges to 1 as $n \to \infty$. Multiply and divide the left hand side of (7) by $(j/n)^{r-1}$. Since $r \le j \ll n$ and $1 - 1/j$ is increasing, we have

$$C\left(\frac{n-d-1}{n}\right)^{j-1}\left(1-\frac{1}{r}\right)^{r-1}(1+o(1))^{r-1}\left(\frac{j}{n}\right)^{r-1} \leq E[Y_j].$$

It follows immediately that for $r \leq j \leq m \ll n$, $(j/n)^{r-1} = O\left(E\left[Y_j\right]\right)$. ■

We complete the proof of Theorem 2 using the following identity.

$$E[\overline{X}_m] = \sum\nolimits_{k=r+1}^{m} E[X_k] = \sum\nolimits_{k=r+1}^{m} \frac{1}{n} E[\overline{Y}_{k-1}] = \sum\nolimits_{k=r+1}^{m} \frac{1}{n} \sum\nolimits_{j=r}^{k-1} E[Y_j].$$

The second equality holds since the probability of a rejection is equal to the expected number of saturated cells, divided by the number n of cells. The index k starts at $r+1$ since all calls up to r are accepted. The index j starts at r since at least $r \leq j$ calls are needed for the j-th to saturate a cluster.

By Lemma 9,

$$E[\overline{X}_m] = \frac{1}{n} \sum\nolimits_{k=r+1}^{m} \sum\nolimits_{j=r}^{k-1} \Theta\left(\frac{j}{n}\right)^{r-1} = \Theta\left(\frac{m^{r+1}}{n^r}\right). \tag{8}$$

Consequently,

$$\lim_{n\to\infty} E[\overline{X}_m] = \begin{cases} 0 & \text{if } m \ll n^{\frac{r}{r+1}}, \\ \infty & \text{if } m \gg n^{\frac{r}{r+1}}. \end{cases} \tag{9}$$

This concludes the proof of Theorem 2.

Theorem 2 holds for more general r-uniform hypergraphs than those which satisfy Condition 1, because the inequality

$$r - 1 \leq |\cup A| - |A| \tag{10}$$

was used in our calculations only for sets A of hyperedges with $1 \leq |A| \leq m$. For example, although cycle graphs violate Condition 1, the statement of Theorem 2 holds for large cycle graphs. In general, the statement holds for hypergraphs of large girth; i.e., for a connected r-uniform hypergraph $\mathcal{H} = (V, E)$ with n hyperedges each of which meets at most d others, and such that (10) holds for each non-empty set A of hyperedges with $|A| \leq m = m(n)$.

3.2 Proof of Theorem 4

We spell out for our situation the martingale machinery we need from Alon and Spencer [1], page 89. Let $\Omega = E^{[m]}$ be the probability space of maps $g : [m] \to E$ with the measure $\Pr[g(b) = a] = 1/n$, with the values of g mutually independent. We use the gradation

$$\emptyset = B_0 \subset B_1 \subset \cdots \subset B_m = [m]$$

where we take B_i=[i].

Let $L : E^{[m]} \to \mathbb{R}$ be the random variable $\overline{X}_m$, i.e., the number of rejected calls. As in [1], we define a martingale $Z_0, Z_1, \dots, Z_m$ by

$$Z_i(h) = E[L(g)|g(b) = h(b) \text{ for all } b \in B_i]$$

$Z_0(h) = E[\overline{X}_m]$ and $Z_m(h)$ is the number of rejections of h. L satisfies the Lipschitz condition relative to the gradation if for all $0 \leq i < m$,

$$h, h' \text{ differ only on } B_{i+1} - B_i \Rightarrow |L(h') - L(h)| \leq 1,$$

which is the case for the number of rejections, since changing the value of a map $h : [m] \to E$ at one argument changes its number of rejections by at most one.

With this setup, Azuma's inequality (Theorem 4.2 in [1]) states that for all $\lambda > 0$,

$$\Pr[|\overline{X}_m - E[\overline{X}_m]| > \lambda\sqrt{m}] < 2e^{-\lambda^2/2}.$$

Let $\epsilon > 0$ and put $\lambda = \epsilon E[\overline{X}_m]/\sqrt{m}$. Azuma's inequality becomes

$$\Pr\left[\left|\frac{\overline{X}_m}{E[\overline{X}_m]} - 1\right| > \epsilon\right] < 2e^{-\epsilon^2 E[\overline{X}_m]^2/2m}.$$

Taking $m \gg n^{\frac{r}{r+1/2}}$, the estimate (8) in the proof of Theorem 2 implies that $E[\overline{X}_m]/\sqrt{m} \to \infty$ as $n \to \infty$, which with Azuma's inequality implies that $\overline{X}_m \sim E[\overline{X}_m]$ almost surely.

3.3 Proof of Proposition 6

Let $\mathcal{A} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. A *word* of $\mathcal{A}$ is a finite sequence of elements of $\mathcal{A}$. The *empty word*, denoted by ϵ, is considered a word of $\mathcal{A}$. A *regular expression* r is a pattern for a set of words of $\mathcal{A}$ which are said to *match* r. Regular expressions are built up from words of A using concatenation, union, the Kleene $*$-operation, and difference. If the regular expression r is a word, then r matches only r. If A and B are regular expressions that match words u and v, respectively, then AB matches uv, $(A \cup B)$ matches either u or v, $(A)^*$ matches zero or more occurrences of u, and $A - B$ matches u but not v.

A sequence of calls to a path will fail to be admissible if it assigns three or more calls to some edge of the graph, or if it assigns a sequence of calls to edges that matches $\mathbf{21}^*\mathbf{2}$.

Recall that $A(m,n)$ denotes the number of admissible functions $f : [m] \to [n]$,where $m, n \geq 1$. Denote by w_f the word of length n whose k-th symbol is the number of times that f takes the value k, where $0 \leq k < n$. In particular, the word w_f of an admissible function f is a word in the alphabet $\mathcal{A} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. Let $\mathcal{L} = \{w_f : f : [m] \to [n] \text{ is admissible, where } m, n \geq 1\}$. We use regular expressions [2] to describe subsets of $\mathcal{L}$ that we will need.

We write $A(m,n) = \widehat{M}(m,n) + N(m,n)$, where $\widehat{M}(m,n)$ is the number of admissible functions $f : [m] \to [n]$ whose word w_f does not end in $\mathbf{21}^*$ and where $N(m,n)$ is the number of admissible functions f whose word w_f ends in $\mathbf{21}^*$.

It will be convenient to define $M(m,z) = \widehat{M}(m, m+z)$, where $z \geq 0$. We write a formal sum of words of $\mathcal{L}$ that represents the admissible functions counted by $M(m,z)$ as follows

$$\binom{m+z}{z}\mathbf{1}^m + \sum_{k=1}^{\lfloor m/2 \rfloor} \sum_{s=0}^{m-2k} \sum_{\vec{d} \in D_{s,k}} \sum_{\vec{e} \in E_{s,k}} \binom{s+k+z}{z} \mathbf{1}^{d_0}\alpha_{e_1}\mathbf{1}^{d_1}\cdots\alpha_{e_k}\mathbf{1}^{d_k} \tag{11}$$

where $D_{s,k} = \{(d_0, \ldots, d_k) : d_i \geq 0, \sum_{j=0}^k d_j = s\}$, $E_{s,k} = \{(e_1, \ldots, e_k) : e_i \geq 2, \sum_{j=1}^k e_j = m - s\}$, and where α_i denotes the word $\mathbf{21}^{i-2}\mathbf{0}$, where $i \geq 2$.

It is clear that $\mathcal{L} = (\mathbf{0}^*\mathbf{1}^*(\mathbf{21}^*\mathbf{0})^*)^* \, (\epsilon \cup \mathbf{21}^*)$. Every word $x \in (\mathbf{0}^*\mathbf{1}^*(\mathbf{21}^*\mathbf{0})^*)^*$ comes from a unique word y of the form

$$\mathbf{1}^{d_0}\alpha_{e_1}\mathbf{1}^{d_1}\cdots\alpha_{e_k}\mathbf{1}^{d_k} \tag{12}$$

by inserting $z = |x| - |y|$ $\mathbf{0}$'s into y. We call the word y the associated word of x (and of f). Note that the z $\mathbf{0}$'s are never inserted into any α.

If there is no α occurring in x, then $y = \mathbf{1}^m$. There are $\binom{m+z}{z}$ ways of inserting the z zeros into y to obtain x. Otherwise, at least one α occurs in x. Let k be the number of α's that occur, and let s denote the number of $\mathbf{1}$'s that occur. There are $\binom{s+k+z}{z}$ ways of inserting the z zeros into y to obtain x. This shows that formal sum of words (11) represents the functions counted by $M(m,z)$.

It remains to compute the number of functions $f : [m] \to [m+z]$ that have an associated word of the form (12).

Let $f : [m] \to [n]$ be a finite function with word w_f, let $\#w_f$ denote the number of functions $g : [m] \to [n]$ with $w_g = w_f$. For j with $1 \leq j \leq n$ let $|f^{-1}(j)|$ denote the cardinality of the pre-image of j. Then

$$\#w_f = \binom{m}{|f^{-1}(1)|, \ldots, |f^{-1}(n)|}. \tag{13}$$

Furthermore, if $f : [j] \to [r]$ and $g : [k] \to [s]$ are finite functions with words $u = w_f$ and $v = w_g$ respectively, then

$$\#(uv) = \#u \cdot \#v \binom{C(u) + C(v)}{C(u)} = \#u \cdot \#v \binom{j+k}{j} \tag{14}$$

where $C(w)$ is the sum of the digits occurring in w.

Now we can evaluate (11). The first term of $M(m,z)$ equals

$$\binom{m+z}{z}\#(\mathbf{1}^m) = \binom{m+z}{z} m! = \frac{(m+z)!}{z!}.$$

The second term is

$$\sum_{k=1}^{\lfloor m/2\rfloor}\sum_{s=0}^{m-2k}\sum_{\vec{d}\in D_{s,k}}\sum_{\vec{e}\in E_{s,k}}\binom{s+k+z}{z}\#(\mathbf{1}^{d_0}\alpha_{e_1}\mathbf{1}^{d_1}\cdots\alpha_{e_k}\mathbf{1}^{d_k})$$

$$=\sum_{k=1}^{\lfloor m/2\rfloor}\sum_{s=0}^{m-2k}\sum_{\vec{d}\in D_{s,k}}\sum_{\vec{e}\in E_{s,k}}\binom{s+k+z}{z}\#(\mathbf{1}^{d_0})\#(\alpha_{e_1})\cdots\binom{m}{d_0,e_1,\cdots,d_k}$$

$$=\sum_{k=1}^{\lfloor m/2\rfloor}\sum_{s=0}^{m-2k}\sum_{\vec{d}\in D_{s,k}}\sum_{\vec{e}\in E_{s,k}}\binom{s+k+z}{z}\frac{d_0!e_1!\cdots d_k!}{2^k}\binom{m}{d_0,e_1,\cdots,d_k}$$

$$=m!\sum_{k=1}^{\lfloor q/2\rfloor}\frac{1}{2^k}\sum_{s=0}^{q-2k}\binom{s+k+z}{s,k,z}\binom{m-s-k-1}{k-1}.$$

We next compute $N(m,n)$.

Proposition 10

$$N(m,n)=\frac{1}{2}\sum\nolimits_{k=1}^{m-1}M(m-k-1,n-m+1)(m-k)_{k+1}. \tag{15}$$

Proof. Suppose that $f:[m]\to[n]$ has word w_f ending in $\mathbf{21}^{k-1}$ where $1\le k\le m-1$. Then $w_f=uv$, where u does not end with $\mathbf{21}^*$. Furthermore $|u|=n-k$, and $C(u)=m-k-1$. The number of functions $g:[m-k-1]\to[n-k]$ with word $w_g=u$ is by definition $M(m-k-1,n-m+1)$. By (13), the number of functions $h:[k+1]\to[k]$ with $w_h=v$ is $\frac{(k+1)!}{2}$. By (14), the number of functions f is

$$\sum_{k=1}^{m-1}M(m-k-1,n-m+1)\frac{(k+1)!}{2}\binom{m}{k+1}.$$

■

3.4 Proof of Proposition 7

Proof. Applying Gosper's algorithm [5, 6] to the inner sum

$$\sum\nolimits_{s=0}^{m-2k}\binom{s+k+z}{s,k,z}\binom{m-s-k-1}{k-1}$$

of $M(m,z)$ one obtains

$$\frac{\csc(k\pi-m\pi)\Gamma(-2k-z)\Gamma(1+k+z)\sin(k\pi)}{\Gamma(1-2k+m)\Gamma(-m-z)}.$$

Apply the reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$$

to the preceding and remove singularities to obtain

$$M(m,z) = (z+1)_m + m! \sum_{k=1}^{\lfloor m/2 \rfloor} \frac{1}{2^k} \binom{k+z}{z} \binom{m+z}{m-2k}.$$

Taking $z = n-m$ and applying Gosper's Algorithm yields the following.

$$M(m,n-m) = (n-m+1)_m \cdot {}_3F_2 \left[\begin{matrix} n-m+1, -m/2, -(m-1)/2 \\ (n-m+1)/2, (n-m+2)/2 \end{matrix} ; \frac{1}{2} \right]. \tag{16}$$

By applying Gosper's algorithm to the expression (15) for $N(m,n)$, one obtains the identity for the number of admissible functions $f : [m] \to [n]$. ■

3.5 Proof of Theorem 5

Our proof of the limit probability distribution depends on the following asymptotic formula.

Lemma 11 *Let $m = m(n) \ll n$. Then*

$$\log\left(\Pr[f : [m] \to [n] \text{ is admissible}]\right) \sim -n \sum_{k=1}^{\infty} \frac{(m/n)^{2k+1}}{(2k)(2k+1)}. \tag{17}$$

Proof. The theorem follows easily from this lemma. Let $s(m,n)$ denote the right hand side of formula (17). Taking $m = An^{2/3}$ where $A > 0$, we have $s(An^{2/3}, n) = -A^3/6 + O(n^{-2/3})$. If $m \ll n^{2/3}$, then for each $A > 0$, there exists an integer N_A so that for all $n > N_A$, $m < An^{2/3}$, and hence $s(An^{2/3}, n) < s(m,n) \leq 0$. Consequently, $e^{-A^3/6} \leq \liminf_{n\to\infty} e^{s(m,n)} \leq 1$. Since A is arbitrary, let A tend to 0. Conversely, if $m \gg n^{2/3}$, then for any $A > 0$, $0 \leq \limsup_{n\to\infty} e^{s(m,n)} \leq e^{-A^3/6}$. This time let A tend to ∞. ■

Recall that

$$A(m,n) = M(m,n-m) + N(m,n) \tag{18}$$

We will show that for $m \ll n$, $n^{-m}N(m,n) \to 0$, so that the probability of admissibility is $n^{-m}M(m,n-m)$. The proof of the lemma will use an asymptotic equivalence that reduces the ${}_3F_2$ term occurring in the hypergeometric identity (16) for $M(m,n-m)$ to a more manageable ${}_2F_1$ term.

Claim Define φ, ψ by

$$\varphi(m,n) = {}_2F_1 \left[\begin{matrix} -m/2, -(m-1)/2 \\ (n-m+2)/2 \end{matrix} ; 1 \right] \tag{19}$$

$$\psi(m,n) = {}_3F_2 \left[\begin{matrix} n-m+1, -m/2, -(m-1)/2 \\ (n-m+1)/2, (n-m+2)/2 \end{matrix} ; \frac{1}{2} \right]. \tag{20}$$

Then, for $m \ll n$,

$$n^{-m}(n-m+1)_m\varphi(m,n) \sim n^{-m}(n-m+1)_m\psi(m,n).$$

Define

$$a_k(m,n) = \frac{(n-m+1)_k(-m/2)_k(-(m-1)/2)_k}{((n-m+1)/2)_k(n-m+2)/2)_k}\frac{1}{2^k k!},$$
$$b_k(m,n) = \frac{(-m/2)_k(-(m-1)/2)_k}{((n-m+2)/2)_k}\frac{1}{k!}.$$

Then $\psi(m,n) = \sum_{k=0}^{\infty} a_k(m,n)$ and $\varphi(m,n) = \sum_{k=0}^{\infty} b_k(m,n)$. Note that for $m \ll n$,

$$\frac{a_{\lfloor\sqrt{m}\rfloor}(m,n)}{b_{\lfloor\sqrt{m}\rfloor}(m,n)} = \prod_{j=0}^{\lfloor\sqrt{m}\rfloor-1}\frac{1+\frac{-m+1+j}{n}}{1+\frac{-m+1+2j}{n}} = \exp\left(-\frac{\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)}{2n} + O\left(\frac{m}{n}\right)^2\right).$$

Write $a_k = a_k(m,n)$, $b_k = b_k(m,n)$. Let $\epsilon > 0$. Choose n so large that $m/n < \epsilon$. Then for each k with $0 \le k \le \lfloor\sqrt{m}\rfloor$,

$$1-\epsilon < \frac{a_{\lfloor\sqrt{m}\rfloor}(m,n)}{b_{\lfloor\sqrt{m}\rfloor}(m,n)} \le \frac{a_k}{b_k} \le 1$$

since $a_k(m,n)/b_k(m,n)$ is decreasing with k. Consequently,

$$0 \le (1-\epsilon)\sum_{k=0}^{\lfloor\sqrt{m}\rfloor} b_k \le \sum_{k=0}^{\lfloor\sqrt{m}\rfloor}\frac{a_k}{b_k}b_k = \sum_{k=0}^{\lfloor\sqrt{m}\rfloor} a_k \le \sum_{k=0}^{\lfloor\sqrt{m}\rfloor} b_k.$$

It follows immediately that $\sum_{k=0}^{\lfloor\sqrt{m}\rfloor} a_k(m,n) \sim \sum_{k=0}^{\lfloor\sqrt{m}\rfloor} b_k(m,n)$.

Let $c(m,n) = n^{-m}(n-m+1)_m$. Define $\widehat{\varphi}(m,n) = c(m,n)\sum_{k=0}^{\lfloor\sqrt{m}\rfloor} b_k(m,n)$ and $\widehat{\psi}(m,n) = c(m,n)\sum_{k=0}^{\lfloor\sqrt{m}\rfloor} a_k(m,n)$. $\widehat{\varphi}(m,n)$ and $\widehat{\psi}(m,n)$ are asymptotically equivalent. Furthermore, they converge to values in $[0,1]$ for $m \ll n$, because they are non-negative and bounded by $c(m,n)\psi(m,n)$ and $c(m,n)\varphi(m,n)$ respectively, each of which are probabilities. The formula of 7 counts the number of admissible maps, so $c(m,n)\psi(m,n)$ is between 0 and 1, and a computation following will show the same for $c(m,n)\varphi(m,n)$. Since $c(m,n)\varphi(m,n) - \widehat{\varphi}(m,n)$ is the tail of a convergent series of positive terms, we have $c(m,n)\varphi(m,n) \sim \widehat{\varphi}(m,n)$. Similarly, $c(m,n)\psi(m,n) \sim \widehat{\psi}(m,n)$.

Suppose that $m \ll n$. The preceding claim, along with the following identity of Gauss,

$${}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\text{if } \Re(c-a-b) > 0,$$

applied to $n^{-m}M(m,n-m)$ yields

$$n^{-m}(n-m+1)_m\,\psi(m,n) \sim n^{-m}(n-m+1)_m\frac{\Gamma(\frac{n-m}{2}+1)\Gamma(\frac{n+m+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2}+1)}. \quad (21)$$

Let $C = m/n$. Apply the identity $\Gamma(k+\frac{1}{2}) = \sqrt{\pi}2^{-2k}(2k)!/k!$, valid for k with $2k \in \mathbb{Z}$ to (21). We obtain

$$n^{-Cn}\frac{n!}{(n(1-C))!}\frac{(n(1-C)/2)!}{\sqrt{\pi}2^{-n}n!/(n/2)!}\frac{\sqrt{\pi}2^{-n(1+C)}(n(1+C))!/(n(1+C)/2)!}{(n/2)!} \tag{22}$$

$$= (2n)^{-Cn}\frac{(n(1+C))!}{(n(1+C)/2)!}\frac{(n(1-C)/2)!}{(n(1+C))!} \sim e^{-Cn}((1-C)^{C-1}(1+C)^{1+C})^{\frac{n}{2}}, \tag{23}$$

where the asymptotic equivalence follows from Sterling's approximation. Taking logarithms and expanding $\log(1 \pm C)$ in a power series centered at 0, we obtain the desired infinite series (17).

In view of (18), we account for $N(m,n)$.

Lemma 12 *If $m = m(n) \ll n$, then $\lim_{n\to\infty} n^{-m}N(m,n) = 0$.*

Proof.

$$\frac{N(m,n)}{n^m} = \sum_{k=1}^{m-1}\frac{M(m-k-1,n-m+1)}{2n^{m-k-1}}\frac{(m-k)_{k+1}}{n^{k+1}} \leq \sum_{k=1}^{m-1}\frac{(m-k)_{k+1}}{n^{k+1}} \tag{24}$$

since $n^{-(m-k+1)}M(m-k-1,n-m+1) \leq 1$. By Gosper's algorithm[5], (24) $= e^n m(m-1)n^{-m}\Gamma(m-1,n)$, where $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$ is the incomplete Gamma function. Let $0 < \epsilon < 1$, and choose n so large that $m/n < \epsilon$. Using the asymptotic expansion

$$\Gamma(m-1,n) \sim n^{m-2}e^{-n}\sum_{k=0}^{\infty}(-1)^k\frac{(2-m)_k}{n^k},$$

valid for $m \ll n$ [8], it follows that

$$(24) \sim \frac{m(m-1)}{n^2}\left(1+\sum_{k=1}^{m-1}\prod_{j=0}^{k-1}\frac{(m-2-j)}{n}\right) \leq \epsilon^2\sum_{k=0}^{\infty}\epsilon^k \leq 2\epsilon^2.$$

■

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