Nearly equal distances and Szemerédi's regularity lemma

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Abstract

A point set is *separated* if the minimum distance between its elements is one. Two numbers are called *nearly equal* if they differ by at most one. If a fixed positive percentage of all pairs of points belonging to a separated set of size n in $\mathbf{R}^3$ determine nearly equal distances, then the diameter of the set is at least constant times n. This proves a conjecture of Erdős.

1 Introduction

In 1946, Erdős [E46] raised the following problem on *repeated distances* determined by a point set: Given n points in the plane (or, more generally, in $\mathbf{R}^d$), at most how many of the $\binom{n}{2}$ interpoint distances can coincide? It is conjectured that in the plane this maximum is $n^{1+\frac{\text{const}}{\log\log n}}$, which is asymptotically sharp, for example for a $\sqrt{n} \times \sqrt{n}$ piece of the integer lattice. The best known upper estimate is only $O(n^{4/3})$ [SST84], [S97]. In 3-space, the best known upper bound is $n^{3/2}\beta(n)$, where $\beta(n)$ is an extremely slowly increasing function related to the inverse Ackermann function [CEG+90]. However, the truth is probably closer to $n^{4/3}$. In higher dimensions, Lenz construction gives the asymptotically tight answer, which is quadratic in n (e.g. see [PA95], [BMP05]). These questions are intimately related to various problems concerning incidences between points and curves, surfaces, etc. (See [AS02], [PS04].)

Erdős observed that the answer to the above problem does not remain the same if one counts the number of distances that are *nearly equal*, where several distances

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are said to be nearly equal if they differ by at most 1, i.e. they all lie in an interval $[t, t+1]$ for some $t > 0$. Clearly, to exclude trivial examples, one needs to consider only *separated* point sets, i.e., point sets where the minimum distance between two points is at least 1. Erdős et al. [EMPS91] (see also [EMP93], [MPS02]) proved that for any $t > 0$, $d \geq 2$, and for any separated set P of n points in $\mathbf{R}^d$, where n is sufficiently large, the number of point pairs in P whose distance lies in the interval $[t, t+1]$ is at most $T(d, n) = \frac{n^2}{2}(1 - \frac{1}{d} + o(1))$. Here, $T(d, n)$ denotes the number of edges of a balanced d-partite graph on n vertices [B78], which is known to be the maximum number of edges in any graph of n vertices that does not contain a complete subgraph with $d + 1$ vertices. Moreover, this bound can be attained for every $t \geq t(d, n)$, as shown by the following construction (described here only for $d = 3$). Let t be a sufficiently large number, and let v_1, v_2, v_3 be the vertices of an equilateral triangle in the plane $x_3 = 0$, with edge length t. At each v_i draw a line perpendicular to the plane $x_3 = 0$, and on each of these lines pick $\lfloor n/3 \rfloor$ or $\lceil n/3 \rceil$ distinct points whose x_3-coordinates are integers between 0 and $n/3$, so that the total number of points is n (see Figure 1). If t is sufficiently large depending on n (roughly $\frac{1}{18}n^2$), the distance between any pair of points selected on different perpendicular lines belongs to the interval $[t, t+1]$.

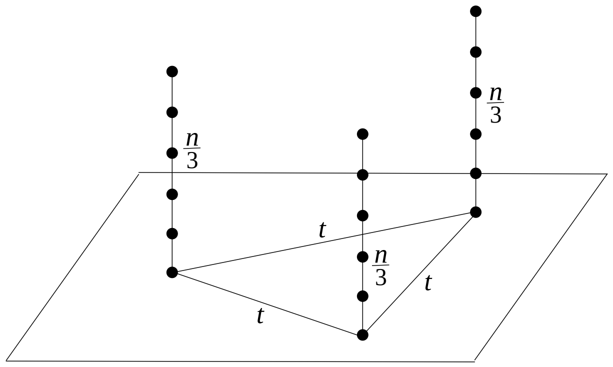

Figure 1. n points in $\mathbf{R}^3$ can determine $\frac{1}{3}n^2$ nearly equal distances.

The question arises, what is the minimal diameter of a separated set of n points in $\mathbf{R}^d$ with $\Omega(n^2)$ nearly equal distances? In the plane the answer is $\Theta(n^2)$, by the Pythagorean theorem. The problem becomes more interesting in three dimensions. Notice that the diameter of the configuration depicted in Figure 1 is $\Omega(n^2)$. However, it is easy to find a set of n points in $\mathbf{R}^3$ with $\frac{n^2}{4}$ nearly equal distances, whose diameter is $O(n)$: Take two $\sqrt{\frac{n}{2}} \times \sqrt{\frac{n}{2}}$ integer grids in two parallel planes at distance $\frac{n}{2}$ from each other (see Figure 2). Erdős conjectured that there is no such example with diameter $o(n)$.

The aim of this note is to prove this conjecture.

Theorem 1.1. *Let $\varepsilon > 0$ be fixed and let P be a separated set of n points in $\mathbf{R}^3$ containing at least εn^2 pairs $(\mathbf{u}, \mathbf{v})$, $\mathbf{u}, \mathbf{v} \in P$, with $||\mathbf{u} - \mathbf{v}|| \in [t, t+1]$ for some fixed real number $t > 0$. Then the diameter of P satisfies $diam(P) = \Omega(n)$.*

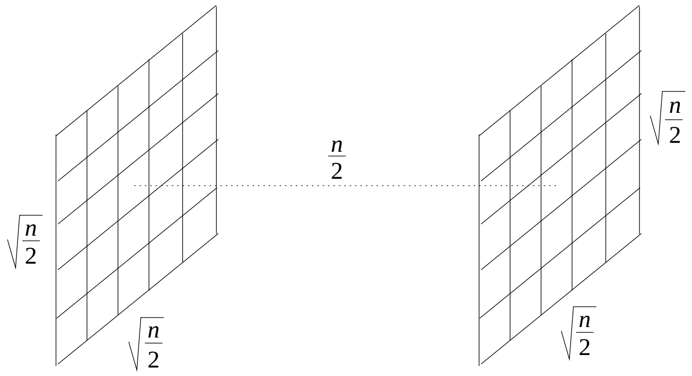

Figure 2. An n-point separated set in $\mathbf{R}^3$ which determines $\frac{1}{4}n^2$ nearly equal distances and has diameter $O(n)$.

The proof is based on Szemerédi's regularity lemma [KS96] and on a Ramsey-type result for dot products of vectors, derived in [APPRS05]. The geometric component of the proof (see Section 3) does not easily generalize to higher dimensions. We will return to the higher-dimensional analogue of Theorem 1.1 in a subsequent paper [PRV05].

2 Using Szemerédi's regularity lemma

In this section, we prove the following theorem.

Theorem 2.1. *Let $\varepsilon > 0$ and let P be a set of n points in $\mathbf{R}^3$ containing at least εn^2 pairs $(\mathbf{u}, \mathbf{v})$, $\mathbf{u}, \mathbf{v} \in P$, with $||\mathbf{u} - \mathbf{v}|| \in [t, t+1]$ for some fixed real number $t > 0$. Then there exists a constant $c := c(\varepsilon) > 0$ and there are two subsets $Q, R \subset P$, such that $|Q| = |R| = cn$ and $||\mathbf{u} - \mathbf{v}|| \in [t, t+1]$ for all $\mathbf{u} \in Q$, $\mathbf{v} \in R$.*

Proof. Let $G = (V(G), E(G))$ be the graph on the vertex set $V(G) := P$ in which two vertices $\mathbf{u}, \mathbf{v} \in V(G)$ are connected by an edge if and only if $||\mathbf{u} - \mathbf{v}|| \in [t, t+1]$. By the assumptions, we have $|E(G)| \geq \varepsilon n^2$.

Before we could state the "degree form" of Szemerédi's regularity lemma (see e.g., [KS96]), we need a definition.

Definition 2.2. *Let $\delta > 0$. Given a graph G and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair (A, B) is δ-regular if for every $X \subset A$ and $Y \subset B$ satisfying $|X| \geq \delta |A|$ and $|Y| \geq \delta |B|$, we have $|d(X,Y) - d(A,B)| < \delta$.*

Here, d stands for the standard density function $d(X,Y) = \frac{|E(X,Y)|}{|X|\cdot|Y|}$, where $E(X,Y)$ denotes the set of edges between X and Y, two disjoint sets of vertices.

Lemma 2.3. *(Szemerédi's regularity lemma) For every $\delta > 0$, there is an $M := M(\delta)$ such that if $G = (V, E)$ is any graph and $\rho \in [0,1]$ is any real number, then there is a partition of the vertex set V into $k+1$ clusters $V_0, V_1, \ldots, V_k$, and there is a subgraph $G' \subset G$ with the following properties:*

1. *$k \le M$,*

2. *$|V_0| \le \delta|V|$,*

3. *all clusters V_i, $i \ge 1$, are of the same size $m \le \lceil \delta|V| \rceil$,*

4. *$deg_{G'}(v) > deg_G(v) - (\rho + \delta)|V|$ for all $v \in V$,*

5. *$E(G'(V_i)) = \emptyset$ for all $i \ge 1$,*

6. *all pairs $G'(V_i, V_j)$, $1 \le i < j \le k$, are δ-regular, each with a density either 0 or greater than ρ.*

Consider the graph G and set parameters $\rho = 0$, $\delta = \min\{\frac{1}{2^{12}}, \frac{2\varepsilon}{5}\}$. Using Lemma 2.3, we obtain a partition $V_0, V_1, \ldots, V_k$ meeting the requirements. Delete all elements in V_0 to obtain a "pure" graph G'' with $\ge (1-\delta)|V|$ vertices, whose vertex set $V(G'') = V(G') \setminus V_0$ is partitioned into k clusters $V_1, V_2, \ldots, V_k$, where $k \le M(\delta)$, $|V_i| = m$ and $E(G''(V_i)) = \emptyset$ for all $i \ge 1$, and all pairs $G''(V_i, V_j)$, $1 \le i < j \le k$, are δ-regular. The pure graph G'' still contains most of the original edges of G. Indeed, we have $deg_{G'}(v) > deg_G(v) - \delta n$ for every v, whence $|E(G')| > |E(G)| - \delta n^2$. To obtain G'', we remove another set of at most $|V_0|n \le \delta n^2$ edges, which implies

$$|E(G'')| > |E(G)| - 2\delta n^2 \ge (\epsilon - 2\delta)n^2.$$

We claim that there is a pair (V_i, V_j) of clusters in G'' with density $d(V_i, V_j) \ge \alpha := 2(\varepsilon - 2\delta)$. Indeed, otherwise, using the fact that $E(G''(V_i)) = \emptyset$, we conclude that $|E(G'')|$ is too small:

$$|E(G'')| \le \binom{k}{2}\alpha m^2 < \frac{\alpha k^2 m^2}{2} \le (\epsilon - 2\delta)n^2.$$

Let (V_i, V_j) be a δ-regular pair with density $d(V_i, V_j) \ge 2(\varepsilon - 2\delta)$. Define maps ω_1, $\omega_2 : V_i \cup V_j \mapsto \mathbf{R}^5$ as follows:

$$\omega_1(\mathbf{u}) = (u_x, u_y, u_z, ||\mathbf{u}||^2 - t^2, 1),$$

$$\omega_2(\mathbf{u}) = (u_x, u_y, u_z, ||\mathbf{u}||^2 - (t+1)^2, 1),$$

$$\omega_1(\mathbf{v}) = (-2v_x, -2v_y, -2v_z, 1, ||\mathbf{v}||^2),$$

$$\omega_2(\mathbf{v}) = (2v_x, 2v_y, 2v_z, -1, -||\mathbf{v}||^2),$$

for all $\mathbf{u} = (u_x, u_y, u_z) \in V_i \subset \mathbf{R}^3$, $\mathbf{v} = (v_x, v_y, v_z) \in V_j \subset \mathbf{R}^3$.

Then, for all $\mathbf{u} \in V_i$, $\mathbf{v} \in V_j$, the edge $\{\mathbf{u}, \mathbf{v}\}$ is in $E(G'')$, that is, $||\mathbf{u} - \mathbf{v}|| \in [t, t+1]$, if and only if $\omega_1(\mathbf{u}) \cdot \omega_1(\mathbf{v}) \ge 0$ and $\omega_2(\mathbf{u}) \cdot \omega_2(\mathbf{v}) \ge 0$.

Recall the following lemma of Alon et al. [APPRS05], which can be proved using a Borsuk-Ulam type result in range searching, due to Yao and Yao [YY85].

Lemma 2.4. [APPRS05] *Let U and V be finite sets of vectors in $\mathbf{R}^d$. Then there are subsets $U' \subset U$ and $V' \subset V$ such that $|U'| \geq \frac{1}{2^{d+1}}|U|$, $|V'| \geq \frac{1}{2^{d+1}}|V|$ and either $\mathbf{u} \cdot \mathbf{v} \geq 0$ for all $\mathbf{u} \in U'$, $\mathbf{v} \in V'$, or $\mathbf{u} \cdot \mathbf{v} < 0$ for all $\mathbf{u} \in U'$, $\mathbf{v} \in V'$.*

Applying Lemma 2.4 to the sets $U := \omega_1(V_i)$ and $V := \omega_1(V_j)$, we obtain two subsets $V_i' \subset V_i$ and $V_j' \subset V_j$ such that $|V_i'| \geq \frac{1}{2^6}|V_i|$, $|V_j'| \geq \frac{1}{2^6}|V_j|$, and either $\omega_1(\mathbf{u}) \cdot \omega_1(\mathbf{v}) \geq 0$ for all $\mathbf{u} \in V_i'$, $\mathbf{v} \in V_j'$, or $\omega_1(\mathbf{u}) \cdot \omega_1(\mathbf{v}) < 0$ for all $\mathbf{u} \in V_i'$, $\mathbf{v} \in V_j'$.

We claim that $\omega_1(\mathbf{u}) \cdot \omega_1(\mathbf{v}) \geq 0$ for all $\mathbf{u} \in V_i'$, $\mathbf{v} \in V_j'$. Indeed, otherwise $||\mathbf{u}-\mathbf{v}|| < t$ holds for all $\mathbf{u} \in V_i'$, $\mathbf{v} \in V_j'$, which implies that $d(V_i', V_j') = 0$. However, by the δ-regularity of the pair (V_i, V_j), we have $d(V_i', V_j') > d(V_i, V_j) - \delta \geq 2(\varepsilon - 2\delta) - \delta \geq 0$, since $\delta = \min\{\frac{1}{2^{12}}, \frac{2\varepsilon}{5}\}$ and $|V_i'| \geq \frac{1}{2^6}|V_i| > \delta|V_i|$, $|V_j'| \geq \frac{1}{2^6}|V_j| > \delta|V_j|$.

Therefore, we have $\omega_1(\mathbf{u}) \cdot \omega_1(\mathbf{v}) \geq 0$ for all $\mathbf{u} \in V_i'$, $\mathbf{v} \in V_j'$.

Next, we apply Lemma 2.4 to the sets $U := \omega_2(V_i')$ and $V := \omega_2(V_j')$, and we obtain two subsets $V_i'' \subset V_i'$ and $V_j'' \subset V_j'$ such that $|V_i''| \geq \frac{1}{2^6}|V_i'|$, $|V_j''| \geq \frac{1}{2^6}|V_j'|$, and either $\omega_2(\mathbf{u}) \cdot \omega_2(\mathbf{v}) \geq 0$ for all $\mathbf{u} \in V_i''$, $\mathbf{v} \in V_j''$, or $\omega_2(\mathbf{u}) \cdot \omega_2(\mathbf{v}) < 0$ for all $\mathbf{u} \in V_i''$, $\mathbf{v} \in V_j''$.

We now claim that $\omega_2(\mathbf{u}) \cdot \omega_2(\mathbf{v}) \geq 0$ for all $\mathbf{u} \in V_i''$, $\mathbf{v} \in V_j''$. Indeed, otherwise $||\mathbf{u} - \mathbf{v}|| > t + 1$ holds for all $\mathbf{u} \in V_i''$, $\mathbf{v} \in V_j''$, and we have $d(V_i'', V_j'') = 0$. However, by the δ-regularity of the pair (V_i, V_j), we obtain $d(V_i'', V_j'') > d(V_i, V_j) - \delta \geq 2(\varepsilon - 2\delta) - \delta \geq 0$, since $\delta = \min\{\frac{1}{2^{12}}, \frac{2\varepsilon}{5}\}$ and $|V_i''| \geq \frac{1}{2^6}|V_i'| \geq \frac{1}{2^{12}}|V_i| \geq \delta|V_i|$, $|V_j''| \geq \frac{1}{2^6}|V_j'| \geq \frac{1}{2^{12}}|V_j| \geq \delta|V_j|$.

Thus, we conclude that $\omega_1(\mathbf{u}) \cdot \omega_1(\mathbf{v}) \geq 0$ and $\omega_2(\mathbf{u}) \cdot \omega_2(\mathbf{v}) \geq 0$ for all $\mathbf{u} \in V_i''$, $\mathbf{v} \in V_j''$. Furthermore, both V_i'' and V_j'' are of size at least $\delta|V_i| = \delta|V_j| \geq \frac{\delta(1-\delta)}{M(\delta)}n$, where $\delta = \min\{\frac{1}{2^{12}}, \frac{2\varepsilon}{5}\}$. Therefore, we have found two subsets $Q := V_i'' \subset P$ and $R := V_j'' \subset P$ such that $|P| = |Q| = c(\varepsilon)n$ and $||\mathbf{u} - \mathbf{v}|| \in [t, t+1]$ for all $\mathbf{u} \in P$, $\mathbf{v} \in Q$. This completes the proof of Theorem 2.1. ☐

3 Proof of Theorem 1.1

In the previous section we have established that there are two sets of points Q, R of size $\Omega(n)$ such that *all* pairwise distances between points in Q and R are in the interval $[t, t+1]$. In the rest of the paper, we conclude that this is impossible unless $t = \Omega(n)$. The proof proceeds in two steps.

1. We prove that R must contain a special configuration, either two points at large distance or three points forming a triangle of large area and small circumradius.

2. We show that this configuration forces Q to be contained in a region of volume $O(t^2/n)$. Since Q requires volume $\Omega(n)$, this yields $t = \Omega(n)$.

Lemma 3.1. *Let $t \leq \frac{n}{64}$ and let R be a separated set of $n \geq 316$ points in $\mathbf{R}^3$ such that $t \leq ||\mathbf{x}|| \leq t+1$ for all $\mathbf{x} \in R$. Then we have $t \geq 2$, and either there are two points $\mathbf{x_1}, \mathbf{x_2} \in R$ such that $||\mathbf{x_1} - \mathbf{x_2}|| > 0.24t$, or there are three points $\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3} \in R$ such that the area of triangle $\{\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}\}$ is at least $\frac{n}{48}$ and the radius of its circumscribed circle is at most $\frac{t}{2}$.*

Proof. Let $\mathcal{B}$ denote the family of n balls of radius $\frac{1}{2}$, each centered at a point of R. Note that the interiors of any two balls in $\mathcal{B}$ are disjoint, since R is separated. Let $A(t)$ denote the spherical annulus $A(t) = \{\mathbf{x} : t - \frac{1}{2} \leq ||\mathbf{x}|| \leq t + \frac{3}{2}\}$. Clearly, $A(t)$ contains $\mathcal{B}$ and its volume is

$$\mathrm{Vol}(A(t)) = \frac{4\pi}{3}\left(\left(t+\frac{3}{2}\right)^3 - \left(t-\frac{1}{2}\right)^3\right) = \pi\left(8t^2 + 8t + \frac{14}{3}\right).$$

Since the balls in $\mathcal{B}$ are pairwise disjoint and each has volume $\pi/6$, the volume of $A(t)$ must be at least $n\pi/6 \geq 316\pi/6$, which implies $t \geq 2$.

Let $\mathbf{x_1}, \mathbf{x_2}$ denote two points in R whose distance $h := ||\mathbf{x_1} - \mathbf{x_2}||$ is maximal. If $h > 0.24t$, we are done. Assume that $h \leq 0.24t$. By a similar volume argument, we obtain that h must be at least $\sqrt{n}/4$. Otherwise, all balls belonging to $\mathcal{B}$ are contained in a sphere of radius $\frac{\sqrt{n}}{4} + \frac{1}{2}$, which intersects $A(t)$ in a region of volume less than $2(\frac{\sqrt{n}}{4} + \frac{1}{2})^2\pi < n\pi/6$, which is a contradiction. Therefore, we obtain

$$h = ||\mathbf{x_1} - \mathbf{x_2}|| \geq \frac{1}{4}\sqrt{n} \geq 2\sqrt{t},$$

which yields

$$\mathbf{x_1} \cdot (\mathbf{x_1} - \mathbf{x_2}) = \frac{1}{2}(||\mathbf{x_1}||^2 - ||\mathbf{x_2}||^2 + ||\mathbf{x_1} - \mathbf{x_2}||^2) \geq \frac{1}{2}(t^2 - (t+1)^2 + 4t) > 0.$$

Similarly, we have $\mathbf{x_2} \cdot (\mathbf{x_1} - \mathbf{x_2}) < 0$. Hence, there is a point $\mathbf{w} = \beta\mathbf{x_1} + (1-\beta)\mathbf{x_2}$, $\beta \in [0,1]$ such that $\mathbf{w} \cdot (\mathbf{x_1} - \mathbf{x_2}) = 0$. Let H be a plane through the origin, orthogonal to vector $\mathbf{w}$. Project the points of R onto H, and let $\phi(\mathbf{x}) \in H$ denote the projection of $\mathbf{x} \in R$. Let $\mathbf{p} = \phi(\mathbf{x_1}), \mathbf{q} = \phi(\mathbf{x_2})$, and note that $||\mathbf{p} - \mathbf{q}|| = ||\mathbf{x_1} - \mathbf{x_2}|| = h$. Without loss of generality, we can assume that $\mathbf{pq}$ is a vertical line in H. Note that all points in R are at most $0.24t$ away from $\mathbf{x_1}$ and $\mathbf{x_2}$, so they project to within $0.24t$ of the origin. Divide the plane between $\mathbf{p}$ and $\mathbf{q}$ into horizontal strips of height 1. Every point in $\phi(R)$ is contained in one of these strips (see Figure 3). In the i-th strip, let l_i and r_i be the horizontal coordinate of the leftmost and rightmost points belonging to $\phi(R)$.

The balls in $\mathcal{B}$ project to disks of radius $\frac{1}{2}$ in the plane H. The area of the region D covered by these disks

$$\mathrm{Area}(D) \leq \sum_{i=1}^{\lceil h \rceil} 2(r_i - l_i + 1).$$

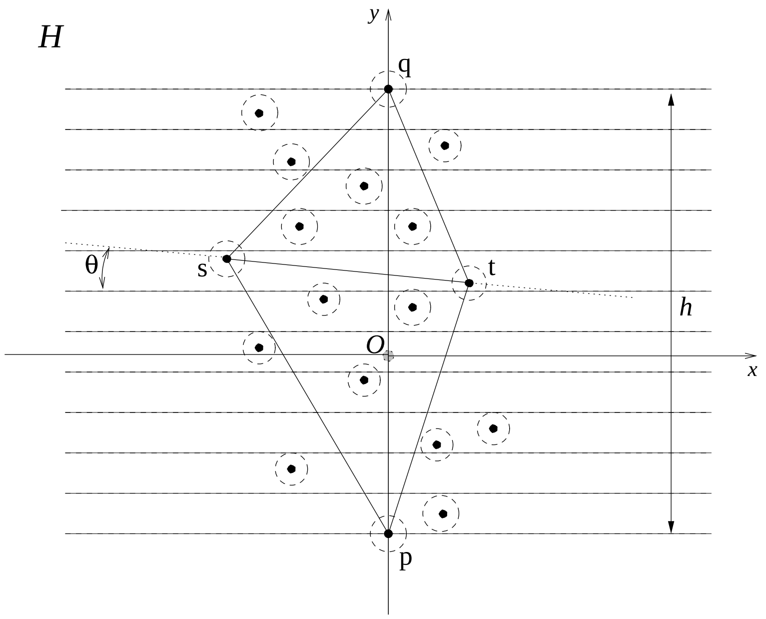

Figure 3. Points of $\phi(R)$ in the plane H, divided into horizontal strips.

Consider a stabbing line through a point of D, perpendicular to the plane H. In view of our choice of H, the distance of the origin from the line is at most $0.24t + \frac{1}{2}$. On each side of H, such a line intersects $A(t)$ in an interval of length at most

$$\sqrt{\left(t+\frac{3}{2}\right)^2 - \left(0.24t+\frac{1}{2}\right)^2} - \sqrt{\left(t-\frac{1}{2}\right)^2 - \left(0.24t+\frac{1}{2}\right)^2},$$

which is less than 2.5, provided that $t \geq 2$. Since the points in R have distance at most $0.24t$ from $\mathbf{x_1}$, they lie in the same halfspace of H as $\mathbf{x_1}$. The balls in $\mathcal{B}$ are contained in the region of $A(t)$ which projects to D, and whose volume is therefore at most 2.5 Area(D). On the other hand, each of these n balls has volume $\pi/6$. Therefore,

$$\frac{\pi n}{6} \leq 2.5 \text{ Area}(D) = 5 \sum_{i=1}^{\lceil h \rceil} (r_i - l_i + 1),$$

and, hence, using $h \geq \sqrt{n}/4 > 4.44$ and $\lceil h \rceil \leq 1.2h$, we deduce that there exists a j such that

$$r_j - l_j + 1 \geq \frac{\pi n}{30 \lceil h \rceil} \geq \frac{\pi n}{36h}.$$

Since $h \leq \frac{t}{4} \leq \frac{n}{256}$, we obtain $\pi n / 36h \geq 22.2$ and

$$r_j - l_j \geq \frac{n}{12h}.$$

Let $\mathbf{s}$ and $\mathbf{t}$ denote the leftmost and righmost points in the j-th horizontal strip, whose coordinates are l_j and r_j, respectively. Let θ denote the angle between the line $\mathbf{st}$ and the x-axis in H (see Figure 3). Then, $\tan\theta \in [-\frac{1}{20}, \frac{1}{20}]$. We have

$$\text{Area}(\{\mathbf{p}, \mathbf{s}, \mathbf{t}\}) + \text{Area}(\{\mathbf{q}, \mathbf{s}, \mathbf{t}\}) = \frac{1}{2} h (r_j - l_j) \geq \frac{n}{24}.$$

Without loss of generality, assume that $\mathrm{Area}(\{\mathbf{p},\mathbf{s},\mathbf{t}\}) \geq \frac{n}{48}$. Then the distance of $\mathbf{p}$ from line $\mathbf{st}$ is at least $\frac{1}{2}h\cos\theta \geq \frac{1}{2}h\sqrt{\frac{400}{401}} > 0.49h$. Set $\mathbf{y_1} := \phi^{-1}(\mathbf{p})$, $\mathbf{y_2} := \phi^{-1}(\mathbf{s})$, and $\mathbf{y_3} := \phi^{-1}(\mathbf{t})$. Since the area cannot increase by projection, we have $\mathrm{Area}(\{\mathbf{y_1},\mathbf{y_2},\mathbf{y_3}\}) \geq \frac{n}{48}$.

Finally, we show that the radius of the circumscribed circle of the triangle $\{\mathbf{y_1},\mathbf{y_2},\mathbf{y_3}\}$ is not too large. Let h' denote the distance of $\mathbf{y_1}$ from the line $\mathbf{y_2y_3}$ and let γ denote the inner angle at $\mathbf{y_3}$. We have $h' > 0.49h$ and $\sin\gamma = h'/||\mathbf{y_1}-\mathbf{y_3}||$. The radius of the circumscribed circle is

$$r = \frac{||\mathbf{y_1}-\mathbf{y_2}||}{\sin\gamma} = \frac{||\mathbf{y_1}-\mathbf{y_2}||\cdot||\mathbf{y_1}-\mathbf{y_3}||}{h'} \leq \frac{h^2}{0.49h} \leq \frac{t}{2}.$$

□

Lemma 3.2. *If* $\mathbf{x_1},\mathbf{x_2} \in \mathbf{R}^3$, $||\mathbf{x_1}-\mathbf{x_2}|| = a$, *then the ring-like region*

$$R(\mathbf{x_1},\mathbf{x_2}) := \{\mathbf{x} : \forall i = 1,2;\ t - \frac{1}{2} \leq ||\mathbf{x}-\mathbf{x_i}|| \leq t + \frac{3}{2}\}$$

has volume at most $4\pi\frac{(2t+1)^2}{a}$.

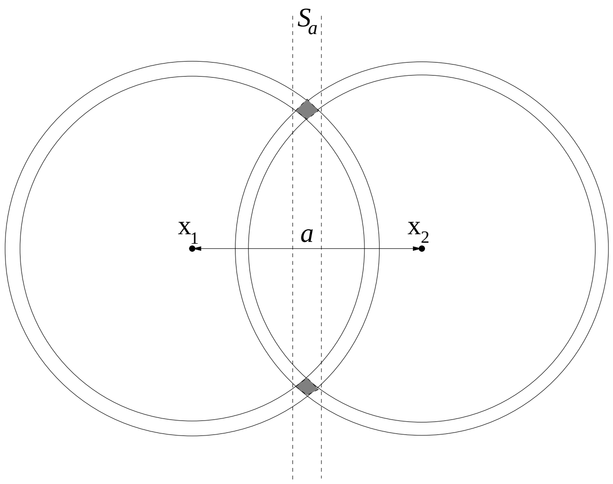

Figure 4. $R(\mathbf{x_1},\mathbf{x_2})$ and the slab S_a.

Proof. We claim that $R(\mathbf{x_1},\mathbf{x_2})$ is contained in the parallel slab S_a (bounded by two parallel planes) of thickness $2(2t+1)a$, orthogonal to line $\mathbf{x_1x_2}$ (see Figure 4). Indeed, assume $\mathbf{x_1} = -\mathbf{x_2}$, $||\mathbf{x_1}|| = ||\mathbf{x_2}|| = \frac{a}{2}$, and $||\mathbf{x}-\mathbf{x_1}||^2, ||\mathbf{x}-\mathbf{x_2}||^2 \in [(t-\frac{1}{2})^2, (t+\frac{3}{2})^2]$. Then,

$$\big|||\mathbf{x}-\mathbf{x_1}||^2 - ||\mathbf{x}-\mathbf{x_2}||^2\big| = 2|\mathbf{x}\cdot(\mathbf{x_2}-\mathbf{x_1})| \leq 4t+2,$$

which means that $\mathbf{x}$ can deviate from the plane of symmetry between $\mathbf{x_1}$ and $\mathbf{x_2}$ by at most $\frac{2t+1}{a}$.

Consider the annulus $A(t)$ between the spheres of radius $t-\frac{1}{2}$ and $t+\frac{3}{2}$, centered at $\mathbf{x_1}$. Take the intersection of this annulus with S_a. The planes of S_a are at distances $l=\frac{a}{2}-\frac{2t+1}{a}$ and $r=\frac{a}{2}+\frac{2t+1}{a}$ from $\mathbf{x_1}$. The resulting volume is

$$\mathrm{Vol}(R(\mathbf{x_1},\mathbf{x_2})) \le \mathrm{Vol}(A(t)\cap S_a)$$
$$\le \int_l^r \left(\pi\left(\left(t+\frac{3}{2}\right)^2 - x^2\right) - \pi\left(\left(t-\frac{1}{2}\right)^2 - x^2\right)\right) dx$$
$$= \int_l^r 2\pi(2t+1)dx = \frac{4\pi(2t+1)^2}{a}.$$

□

Lemma 3.3. *Let $t\ge 2$, and let $\{\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}\}$ be a triangle of area $A\ge 2t+1$ whose sides are of length at most $\frac{t}{4}$ and whose circumscribing circle has radius at most $\frac{t}{2}$. Then the region*

$$R(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) = \{\mathbf{x} : \forall i=1,2,3;\ t-\frac{1}{2}\le ||\mathbf{x}-\mathbf{x_i}|| \le t+\frac{3}{2}\}$$

has volume at most $\frac{16(2t+1)^2}{A}$.

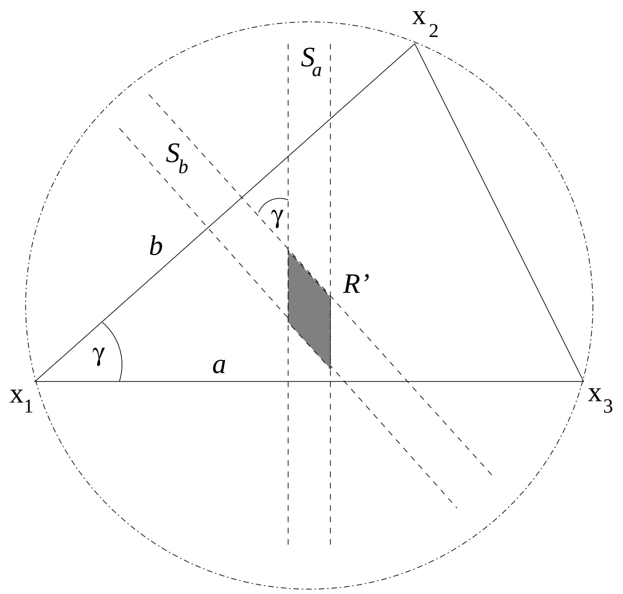

Figure 5. R', the intersection of the infinite prism $S_a\cap S_b$ with the plane of the triangle $\{\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}\}$.

Proof. Consider two points $\mathbf{x_1},\mathbf{x_2}$ at distance a and the region

$$R(\mathbf{x_1},\mathbf{x_2}) = \{\mathbf{x} : \forall i=1,2;\ t-\frac{1}{2}\le ||\mathbf{x}-\mathbf{x_i}|| \le t+\frac{3}{2}\}.$$

As before, $R(\mathbf{x_1},\mathbf{x_2})$ is contained in a slab S_a of thickness $\frac{2(2t+1)}{a}$, orthogonal to line $\mathbf{x_1}\mathbf{x_2}$. The same holds for $R(\mathbf{x_1},\mathbf{x_3})$, that is, if $||\mathbf{x_1}-\mathbf{x_3}||=b$, then $R(\mathbf{x_1},\mathbf{x_3})$ is contained in a slab S_b of thickness $\frac{2(2t+1)}{b}$. Let γ denote the angle between $\mathbf{x_1x_2}$ and $\mathbf{x_1x_3}$. Then the angle between the normal vectors to S_a and S_b is also γ. The area A of the triangle $\{\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}\}$ is $A=\frac{1}{2}ab\sin\gamma$.

The intersection $S_a\cap S_b$ is an infinite prism which intersects the plane of the triangle in a parallelogram R' (see Figure 5). The sides of R' are of length $\frac{2(2t+1)}{a\sin\gamma}$ and $\frac{2(2t+1)}{b\sin\gamma}$, and the angle between them is γ, which implies

$$\mathrm{Area}(R')=\frac{4(2t+1)^2}{ab\sin\gamma}=\frac{2(2t+1)^2}{A}.$$

The center of R' is the center of the circle circumscribed around $\{\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}\}$. Since the radius of the circumscribing circle is at most $\frac{t}{2}$, the distance between any point in R' and $\mathbf{x_1}$ is at most

$$\frac{t}{2}+\frac{2t+1}{a\sin\gamma}+\frac{2t+1}{b\sin\gamma}=\frac{t}{2}+\frac{2t+1}{2A}(b+a)\le\frac{t}{2}+\frac{a+b}{2}\le\frac{3t}{4},$$

where the first inequality follows from the assumption that $A\ge 2t+1$. This implies that the prism $S_a\cap S_b$ intersects the annulus $A(t)$ centered at $\mathbf{x_1}$ within distance $\frac{3t}{4}$ from $\mathbf{x_1}$. Any line within distance $\frac{3t}{4}$ from the center intersects $A(t)$ in two intervals of length at most

$$\sqrt{\left(t+\frac{3}{2}\right)^2-\left(\frac{3t}{4}\right)^2}-\sqrt{\left(t-\frac{1}{2}\right)^2-\left(\frac{3t}{4}\right)^2},$$

which is smaller than 4 for $t\ge 2$. Therefore, we have

$$\mathrm{Vol}(R(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}))\le\mathrm{Vol}(A(t)\cap S_a\cap S_b)\le 8\ \mathrm{Area}(R')=\frac{16(2t+1)^2}{A}.$$

□

Theorem 3.4. *Let Q and R be two separated sets of points in $\mathbf{R}^3$, each of size $n\ge 316$, such that $t\le\|\mathbf{x}-\mathbf{y}\|\le t+1$ for all $\mathbf{x}\in Q$, $\mathbf{y}\in R$. Then, $t\ge\frac{n}{800}$.*

Proof. Suppose that $n\ge 316$ and $t<\frac{n}{800}$. Assume that one of the points in Q is the origin. Then Lemma 3.1 implies that $t\ge 2$, and either there exist two points $\mathbf{x_1},\mathbf{x_2}\in R$ at distance at least $0.24t$, or there exist three points $\mathbf{y_1},\mathbf{y_2},\mathbf{y_3}\in R$ such that the triangle $\{\mathbf{y_1},\mathbf{y_2},\mathbf{y_3}\}$ has area at least $\frac{n}{48}$, its edges are of length at most $\frac{t}{4}$, and its circumradius is at most $\frac{t}{2}$. In the first case, Lemma 3.2 implies that the volume of $R(\mathbf{x_1},\mathbf{x_2})$ is at most $\frac{60(2t+1)^2}{t}$. In the second case, Lemma 3.3 implies that the volume of $R(\mathbf{y_1},\mathbf{y_2},\mathbf{y_3})$ is at most $\frac{768(2t+1)^2}{n}\le\frac{(2t+1)^2}{t}$. In either case, the region must contain a ball of radius $\frac{1}{2}$ around each point of Q, and these balls are pairwise disjoint. Therefore, the volume of the region must be at least $\frac{n\pi}{6}$. For $2\le t<\frac{n}{800}$ this number is at most $\frac{60(2t+1)^2}{t}\le 375t<\frac{n\pi}{6}$, which is a contradiction. □

Now we are in a position to prove Theorem 1.1. Let P be a separated n-point set in $\mathbf{R}^3$ such that there exist εn^2 pairs $(\mathbf{u}, \mathbf{v})$, $\mathbf{u}, \mathbf{v} \in P$, $\varepsilon > 0$, with $||\mathbf{u} - \mathbf{v}|| \in [t, t+1]$ for some fixed real number $t > 0$. Then, by Theorem 2.1, there exist a constant $c := c(\varepsilon) > 0$ and two subsets $Q, R \subset P$ such that $|Q| = |R| = cn$ and $||\mathbf{u} - \mathbf{v}|| \in [t, t+1]$ for all $\mathbf{u} \in Q$, $\mathbf{v} \in R$. Then, by Theorem 3.4, we obtain $t \geq \frac{cn}{800}$. Hence, we have $\mathrm{diam}(P) = \Omega(n)$, as required.

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