Disjoint edges in topological graphs *

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Abstract

A topological graph is a graph drawn in the plane by simple Jordan arcs. It is called simple, if any two edges have at most one point in common. We show that a simple topological graph with no k pairwise disjoint edges has $O\left(n \log^{4k-8} n\right)$ edges. We also show that we cannot drop the simplicity condition, we show a complete topological graph where any two edges have either one or two common points.

1 Introduction

A *topological graph* is a graph drawn in the plane so that its vertices are represented by points in the plane and its edges by simple (non-selfintersecting) Jordan arcs connecting the corresponding points and not passing through any vertex other than its endpoints.

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We will make no notational distinction between the vertices (edges) of the underlying abstract graph, and the points (arcs) representing them in the plane.

It is well known that if a topological graph with n vertices has no pair of crossing edges, then its number of edges cannot exceed $3n-6$. It is conjectured that for every fixed k the maximum number of edges that a topological graph of n vertices can have without containing k pairwise crossing edges is $O(n)$.

Theorem A ([AAPPS97]) *Every simple topological graph of n vertices and no three pairwise crossing edges has $O(n)$ edges.*

Theorem B ([PRT03]) *Every topological graph of n vertices and no three pairwise crossing edges has $O(n)$ edges.*

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For $k > 3$ the best known results are the following.

Theorem C ([PSS96]) *For any $k > 3$, every simple topological graph of n vertices and no k pairwise crossing edges has $O\left(n \log^{2(k-3)} n\right)$ edges.*

Theorem D ([PRT03]) *For every $k > 3$, every topological graph of n vertices and no k pairwise crossing edges has $O\left(n \log^{4(k-3)} n\right)$ edges.*

A topological graph is x-monotone, if all of its edges are x-monotone curves.

Theorem E ([V98]) *For every $k > 3$, every simple x-monotone topological graph of n vertices and no k pairwise crossing edges has $O(n \log n)$ edges.*

In the present note we establish analogous results for *pairwise disjoint* edges, instead of pairsiwe crossing edges.

Theorem 1 *For any $k \geq 2$, every simple topological graph of n vertices and no k pairwise disjoint edges has $O\left(n \log^{4k-8} n\right)$ edges.*

We also prove that here we cannot drop or relax the assumption that the topological graph is simple. If the edges are allowed to cross twice, then we can draw even the complete graph without two disjoint edges.

Theorem 2 *For any n, K_n can be drawn in the plane such that any two edges have one or two points in common.*

2 Proofs of Theorems 1 and 2

A *cut* $e(V_1, V_2)$ of a graph G is the set of edges connecting V_1 and V_2, for any partition of the vertices of G into two parts, V_1 and V_2. The size of the cut $|e(V_1, V_2)|$ is the number of edges in the cut. The *bisection width* $b(G)$ is the size of the minimal cut $|e(V_1, V_2)|$ where $|V_1|, |V_2| \geq |V|/3$. The *edge expansion* of a graph G is

$$\beta(G) = \min_{A \subseteq V} \frac{e(A, V \setminus A)}{\min\{|A|, |V \setminus A|\}}.$$

An *embedding* of a graph H in a graph G maps vertices of H to different vertices of G, and each edge of H to a path of G between the corresponding vertices. The *congestion* of an embedding is the maximum number of paths passing through an edge of G.

Clearly, for any graph G, $\beta(G) \leq 3b(G)/n$. On the other hand, it is possible that $\beta(G)$ is small (even 0) but $b(G)$ is large. However, the following result of Kolman and Matoušek holds.

Theorem F (Kolman, Matoušek [KM03]) *Every graph G has a subgraph H of at least $\frac{2n}{3}$ vertices such that $\beta(H) \geq \frac{b(G)}{n}$.*

Theorem G (Kolman, Matoušek [KM03]) *Let G be a graph on n vertices. Then there exists an embedding of K_n in G with congestion $O\left(\frac{n \log n}{\beta(G)}\right)$.*

The *crossing number* of a graph G is the minimum number of crossing points in any drawing of G. The *pairwise crossing number* (resp. the *odd-crossing number*) of G is the minimum number of pairs of edges that cross (resp. cross an odd number of times) over all drawings of G. The *odd-crossing number* $\text{ODD-CR}(G)$ is the minimum number of pairs of edges that cross an odd number of times, over all drawings of G. For any graph G, let

$$\text{SSQD}(G) = \sum_{v \in V(G)} \text{DEG}(v)^2.$$

Lemma 1 *For every graph G,*

$$\text{ODD-CR}(G) \geq \Omega\left(\frac{b(G)^2}{\log^2 n}\right) - O\left(\text{SSQD}(G)\right).$$

A result analogous to Lemma 1, for the pairwise crossing number instead of the odd-crossing number has been proved by Kolman and Matoušek. Since $\text{ODD-CR}(G) \leq \text{PAIR-CR}(G)$ for every graph, Lemma 1 is an improvement of that result. Our proof is very similar to the original proof of Kolman and Matoušek.

Proof. Let H be a subgraph of G that satisfies the conditions of Theorem F. Since $\text{ODD-CR}(G) \geq \text{ODD-CR}(H)$, it is enough to show that

$$\text{ODD-CR}(H) \geq \Omega\left(\frac{n^2\beta(H)^2}{\log^2 n}\right) - O\left(\text{SSQD}(H)\right).$$

Let m be the number of vertices of H, we have $n \geq m \geq \frac{2n}{3}$. Take a drawing of H that has exactly $\text{ODD-CR}(H)$ pairs of edges that cross an odd number of times. By Theorem G, there exists an embedding of K_m in G with congestion $O\left(\frac{m \log m}{\beta(H)}\right)$. This embedding gives a drawing of K_m. Crossings in this drawing come from edge-crossings of H, and from crossings near the vertices of H. Consider two edges, e_1 and e_2, of K_m, in the present embedding. They correspond to two paths, p_1 and p_2 of H. If p_1 and p_2 cross an odd number of times, then either (i) one of their crossings is near a vertex of H, or (ii) each of p_1 and p_2 contain an edge of H, that cross each other an odd number of times. The number of pairs of edges of K_m that belong to case (i) is $O\left(\text{SSQD}(H)\frac{n^2\log^2 n}{\beta(H)^2}\right)$, and the number of pairs of edges of K_m that belong to case (ii) is $O\left(\text{ODD-CR}(H)\frac{n^2\log^2 n}{\beta(H)^2}\right)$.

On the other hand, it is known ([PT00]) that $\text{ODD-CR}(K_m) = \Omega(m^4)$, therefore,

$$\Omega(m^4) \le O\left((\text{SSQD}(H) + \text{ODD-CR}(H))\frac{n^2\log^2 n}{\beta(H)^2}\right),$$

and using that $m \ge \frac{2n}{3}$, the result follows. $\square$

Lemma 2. *For any $k \ge 2$, every topological graph G of n vertices with no k edges that pairwise cross each other an odd number of times, has $O\left(n\log^{4k-8} n\right)$ edges.*

Proof. We follow the idea of the proofs of Theorems C and D, the proof is by double induction on n and k. For $k = 2$ and any n, the statement follows from an old result of Hanani [H34] that $\text{ODD-CR}(G) = 0$ if and only if G is planar.

Assume that we have already proved Lemma 2 for some $k \ge 2$ and all n. Let n a fixed number and suppose that the lemma holds for $k+1$ and for all topological graphs having fewer than n vertices.

Let G be a topological graph of n vertices with no $k+1$ edges that pairwise cross each other an odd number of times. For simplicity, we denote the underlying abstract graph also by G. For any edge $e \in E(G)$, let $G_e \subset G$ denote the topological graph consisting of all edges of G that cross e an odd number of times. Clearly, G_e has no k edges that pairwise cross each other an odd number of times. Thus, by the induction hypothesis, we have

$$\text{ODD-CR}(G) \le \frac{1}{2}\sum_{e\in E(G)} |E(G_e)| \le \frac{1}{2}|E(G)|O\left(n\log^{4k-8} n\right).$$

Using the fact that $\text{SSQD}(G) \le 2|E(G)|n$ holds for every graph G, Lemma 1 implies that

$$b(G) = O\left(c(\log n)^2|E(G)|n\log^{4k-8} n\right)^{1/2}.$$

Consider a partition of $V(G)$ into two parts of sizes $n_1, n_2 \le 2n/3$ such that the number of edges running between them is $b(G)$. Obviously, the subgraphs induced by these parts have no $k+1$ edges that pairwise cross each other an odd number of times. Apply the induction hypothesis for both parts and obtain

$$|E(G)| \le O\left(n_1\log^{4k-8} n_1\right) + O\left(n_2\log^{4k-8} n_2\right) + b(G).$$

Comparing the last two inequalities, the result follows by some calculation. $\square$

Proof of Theorem 1. Let G be a simple topological graph that has no k pairwise disjoint edges. Let H be a *bipartite* topological subgraph of G with at least half of the original edges. Clearly, H is has no k pairwise disjoint edges either. Let V_1 and V_2 be the two vertex classes of H, that is, all edges are between V_1 and V_2.

Modify the drawing in such a way that (i) we do not loose or introduce any crossing, (ii) we move all vertices in V_1 above the line $y = 1$, and all vertices in V_2 below the line $y = 0$. (iii) each piece of each edge in the strip $0 \leq y \leq 1$ is a vertical segment.

Now replace the part of the drawing above $y = 1$ by its reflection about the y-axis. Erase the part of the drawing in the strip $0 \leq y \leq 1$, and reconnect the corresponding pairs of points on the lines $y = 0$ and $y = 1$, respectively, by straight line segments. This way we obtain another drawing of G. Let e_1 and e_2 be two independent edges of G. It is easy to see that e_1 and e_2 are disjoint in the original drawing if and only if they cross an odd number of times in the new drawing, and they cross in the original drawing if and only if they cross an even number of times in the new drawing. Now the result follows directly from Lemma 2. $\square$

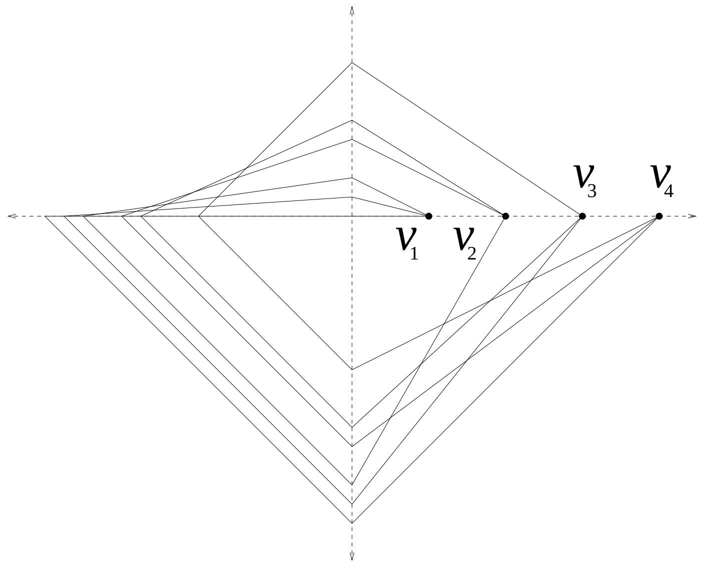

Figure 1.

A drawing of K_4 such that any two edges have one or two points in common.

Proof of Theorem 2 Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n. For $1 \leq i \leq n$, place v_i at $(i, 0)$. Now, for any $1 \leq i < j \leq n$, draw the edge $v_i v_j$ as a polygon through the following points.

$$(0,i),\ (0,i-j/n),\ (i-j/n-n,0),\ (0,i-j/n-n),\ (0,j).$$

It is easy to check that any two edges have one or two points in common.

Remark. In [PST01] it was shown that a complete, simple topological graph of n vertices has $\Omega\left(\log^{1/6} n\right)$ pairwise disjoint edges. Using Theorem 1, it can be improved as follows. Take a complete, simple topological graph of n vertices and let k denote the maximum number of pairwise disjoint edges. Then $\binom{n}{2} = O\left(n \log^{4k-8} n\right)$, so $k = \Omega\left(\log n / \log\log n\right)$.

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