DEGENERATE CROSSING NUMBERS

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Abstract

Let G be a graph with n vertices and $e \geq 4n$ edges, drawn in the plane in such a way that if two or more edges (arcs) share an interior point p, then they must properly cross one another at p. It is shown that the number of crossing points, counted without multiplicity, is at least constant times e and that the order of magnitude of this bound cannot be improved. If, in addition, two edges are allowed to cross only at most once, then the number of crossing points must exceed constant times $(e/n)^4$.

1 Introduction

Let S be a compact surface with no boundary. Given a graph G with no loops or multiple edges, the *crossing number* of G on S, denoted by $\text{CR}_S(G)$, is the minimum number of edge crossings over all proper drawings of G on S. If S is the sphere (or plane) then we simply write $\text{CR}(G)$. A drawing is *proper* if the vertices and edges of G are represented by points and simple Jordan-arcs in S such that no arc representing an edge passes through a point representing a vertex other than its endpoints. Here we count a k-fold crossing $\binom{k}{2}$ times (or, equivalently, no three edges can pass through the same point). We also assume that between the arcs no tangencies are allowed. See [8] for a survey.

G. Rote, M. Sharir, and others asked what happens if multiple crossings are counted only *once* (equivalently, if several edges are allowed to pass through the same point)? To what extent does this modification effect the notion of crossing number?

Let $\text{CR}^*(G)$ denote the *degenerate crossing number* of G, that is, the minimum number of crossing *points* over all drawings of G, where k-fold crossings are also allowed. Of course, we have

$$\text{CR}^*(G) \leq \text{CR}(G),$$

and the two crossing numbers are not necessarily equal. For example, in the plane Kleitman [2] proved that the crossing number of the complete bipartite graph $K_{5,5}$

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with five vertices in its classes is 16. On the other hand, the degenerate crossing number of $K_{5,5}$ in the plane is at most 15. Another example is depicted in Figure 1.

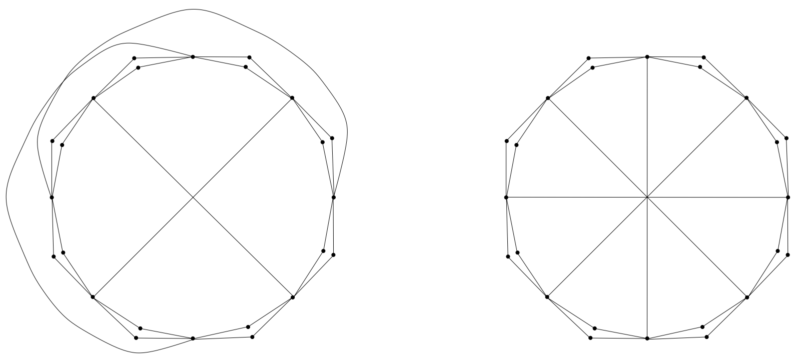

Figure 1: $\text{CR}(G) = 2, \text{CR}^*(G) = 1$.

Let $n = n(G)$ and $e = e(G)$ denote the number of vertices and the number of edges of a graph G. Ajtai, Chvátal, Newborn, Szemerédi [1] and, independently, Leighton [3] proved that

$$\text{CR}(G) \geq \frac{1}{64} \frac{e^3(G)}{n^2(G)}$$

for every graph G with $e(G) \geq 4n(G)$. This statement, which has many interesting applications in combinatorial geometry, easily generalizes to crossing numbers of graphs drawn on any fixed surface S (see [6]).

In the present note we investigate whether the above inequality remains true for the *degenerate* crossing number of G. First, we show that the answer is "no" if we permit drawings in which two edges may cross an arbitrary number of times.

Theorem 1. *Any graph with n vertices and e edges has a proper drawing in the plane with fewer than e crossings, where each crossing point that belongs to the interior of several edges is counted only once. The order of magnitude of this bound cannot be improved if $e \geq 4n$.*

Therefore, in Section 3 we restrict our attention to so-called *simple* drawings, i.e., to proper drawings in which two edges are allowed to cross at most once. From now on, with a slight abuse of notation, $\text{CR}^*(G)$ will stand for the minimum number of crossings over all simple drawings. We prove that in this sense the degenerate crossing number of very "sparse" graphs and very "dense" graphs exceed $\Omega(e^3/n^2)$. More precisely, we have

Theorem 2. *There exists a constant $c^* > 0$ such that the degenerate crossing number of G satisfies*

$$\mathrm{CR}^*(G) \geq c^* \frac{e^4(G)}{n^4(G)},$$

for any graph G with $e(G) \geq 4n(G)$.

If it causes no confusion, in notation and terminology we make no distinction between the graph G and its drawing, and between a vertex (edge) and the point (arc) representing it.

2 Proper drawings with few crossings

In this section, we prove Theorem 1.

Let $\pi = (\pi(1), \pi(2), \ldots, \pi(e))$ be a permutation of the first e positive integers, and let $1 \leq i < j \leq e$. Reversing the order of the elements between $\pi(i)$ and $\pi(j)$, we obtain another permutation

$$\pi' = (\pi(1), \pi(2), \ldots, \pi(i-1), \pi(j), \pi(j-1), \ldots, \pi(i), \pi(j+1), \pi(j+2), \ldots, \pi(e)).$$

Such an operation is called a *swap*.

Lemma 2.1. *Any permutation of e numbers can be obtained from any other permutation by performing at most $e-1$ swaps.*

Proof. The proof is by induction on e. For $e = 1$, the statement is trivial. Suppose that the lemma has been verified for permutations of fewer than e numbers. Let $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(e))$ and $\pi = (\pi(1), \pi(2), \ldots, \pi(e))$ be two permutations of size e. For some j, we have $\pi(j) = \sigma(e)$. To obtain σ from π, we first swap the interval $(\pi(j), \ldots, \pi(e))$ of π. The last element of the resulting permutation $(\pi(1), \pi(2), \ldots, \pi(j-1), \pi(e), \pi(e-1), \ldots, \pi(j))$ is now the same as the last element of the target permutation σ. Proceeding by induction, we can attain using at most $e-2$ further swaps that all elements coincide. $\square$

Proof of Theorem 1. Let G be a graph with e edges and n vertices, $v_1, v_2, \ldots, v_n$. Arbitrarily orient every edge of G. For $1 \leq i \leq n$, place v_i at the point $(0, i)$ on the y-axis. Each edge will be drawn as a continuous arc running close to a huge circle centered at a faraway point of the positive y-axis, so that its initial and final portions are almost horizontal segments, oriented from left to right, that belong to the half-planes $x \geq 0$ and $x \leq 0$, respectively. (See Figure 2.) More precisely, for each edge $\overrightarrow{v_i v_j}$, draw a short almost horizontal initial segment from v_i pointing to the right and a short almost horizontal final segment pointing to v_j from the left. Suppose that all these segments have different slopes. From bottom to top, enumerate the initial segments by $1, 2, \ldots, e$, and assign the same numbers to the final segments of the corresponding edges, lying in the negative half-plane $x \leq 0$. The indices of these final segments (from bottom to top) form a permutation $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(e))$. We have to connect the right endpoint of each initial segment to the left endpoint of the final segment denoted by the same number. These connecting arcs will run parallel to one another, roughly along huge concentric circles, except that at certain points several arcs will cross.

By Lemma 2.1, σ can be obtained from $1, 2, \ldots, e$ by a sequence of at most $e-1$ swaps. We can "realize" each swap as a crossing of the corresponding arcs at a single point. The participating arcs leave the crossing in reverse order. Thus, introducing at most $e-1$ crossings, we can achieve that the order of the connecting arcs is identical to the order in which their final segments must reach the y-axis (from the left).

It follows from Lemma 2.2 (see below) that any proper drawing of G has at least $\frac{e}{3} - n + 2$ crossings. $\square$

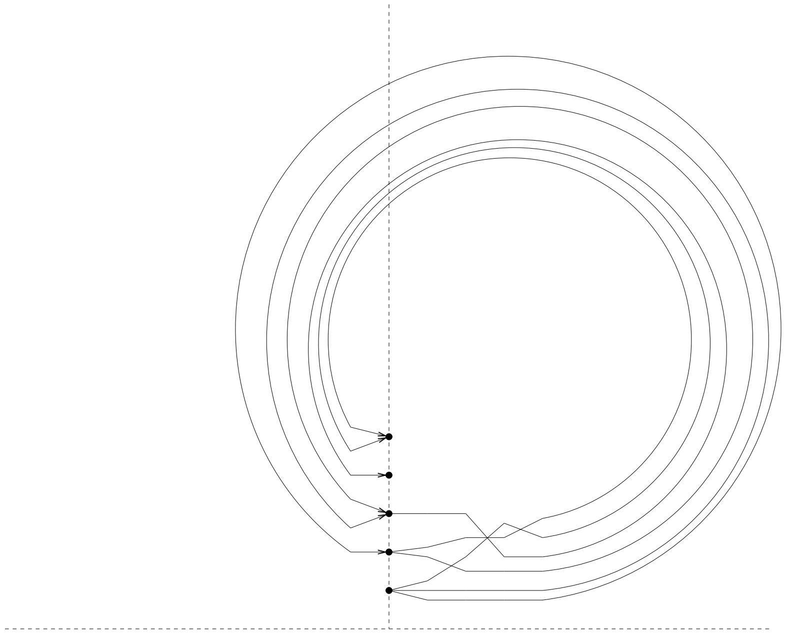

Figure 2: $\text{CR}^*(G) \le e - 1$.

We prove the tightness of Theorem 1 in a slightly more general setting. Let S be a compact surface S with no boundary, whose *Euler characteristic* is χ. That is, we have

$$\chi(S) = \begin{cases} 2 - 2g & \text{if } S \text{ is orientable of genus } g, \\ 2 - g & \text{if } S \text{ is nonorientable of genus } g. \end{cases}$$

Given a connected graph G with no loops or multiple edges, let $\underline{\text{CR}}_S(G)$ stand for the minimum number of crossing points over all *proper* drawings of G on S. Taking the minimum over all *simple* drawings (that is, allowing two edges to cross only at most once), we obtain the *degenerate crossing number* of G on S, denoted by $\text{CR}_S^*(G)$. Clearly, we have $\underline{\text{CR}}_S(G) \leq \text{CR}_S^*(G)$ for any G.

Lemma 2.2. *Let G be a graph with $n(G)$ vertices and $e(G)$ edges, and let S be a surface with Euler characteristic χ. Then we have*

$$\text{CR}_S^*(G) \geq \underline{\text{CR}}_S(G) \geq \frac{e(G)}{3} - n(G) + \chi.$$

Proof. Fix an optimal proper drawing of G on S, i.e., a drawing for which the number of crossings is $\underline{\text{CR}}_S(G)$. Let p be a crossing determined by k edges $e_1, e_2, \ldots, e_k$. Remove from S a small rectangular piece $ABCD$ such that each e_i intersects its boundary in two points $A_i \in AB$ and $C_i \in CD$ and the counterclockwise order of these points is $A_1, A_2, \ldots, A_k, C_1, C_2, \ldots, C_k$. Assume that no further edges of G meet the rectangle $ABCD$. Modify S by adding a *crosscap* at $ABCD$, i.e., by identifying A_i and C_i for every i (and identifying all other "diametrically opposite" pairs of points of the boundary of $ABCD$). In this way, we reduce the number of crossings by one and we obtain a drawing of G on a surface whose Euler characteristic is $\chi(S)-1$. Repeating the same procedure at each crossing, finally we obtain a crossing-free drawing of G on a (nonorientable) surface S' with Euler characteristic $\chi(S) - \underline{\text{CR}}_S(G)$. Let $f(G)$ denote the number of faces in this drawing (embedding). The number of *faces* or *cells* in this embedding is denoted by $f(G)$.

According to *Poincaré's formula*, a generalization of Euler's polyhedral formula, we have

$$n(G) - e(G) + f(G) \geq \chi(S') = \chi(S) - \underline{\text{CR}}_S(G).$$

This inequality becomes an equation, if the embedding is *cellular*, that is, if the boundary of each face is connected. For details, see [4]. Taking into account that $3f(G) \leq 2e(G)$, we obtain

$$\underline{\text{CR}}_S(G) \geq \frac{e(G)}{3} - n(G) + \chi(S),$$

as required. □

3 Simple drawings—Proof of Theorem 2

Let $\text{CR}^*(G)$ stand for the minimum number of crossing points over all *simple* drawings of G in the plane.

Lemma 3.1. *Let G be a graph with n vertices and e edges, and suppose that the crossing number of G satisfies $\text{CR}(G) > 10^3 e(G)n(G)$. Then for the degenerate* *crossing number of G we have*

$$\text{CR}^*(G) \geq \frac{\text{CR}^3(G)}{(40e)^4}.$$

Proof. Consider a simple drawing of G with $\text{CR}^*(G)$ crossing points. Let $M := 40^2e^2/\text{CR}(G)$.

For any crossing (point) p, let $m(p)$ denote the *multiplicity* of p, that is, the number of edges passing through p. Let S denote the set of crossings of multiplicity at most M. For any integer $i \geq 0$, let S_i be the set of crossing points p with $2^iM < m(p) \leq 2^{i+1}M$. Since $m(p)$ cannot exceed $n/2$, we have $S_i = \emptyset$ whenever $2^iM > n/2$. It follows from the generalization of the Szemerédi–Trotter theorem [10], [9] for bounding the number of incidences between a set of points and a set of pseudo-segments that the number of crossings of multiplicity at least k is at most $100\left(\frac{e^2}{k^3} + \frac{e}{k}\right)$. That is,

$$|S_i| \leq 100\left(\frac{e^2}{2^{3i}M^3} + \frac{e}{2^iM}\right)$$

holds for every i. The number of crossing *pairs of edges* is at least $\text{CR}(G)$, and each point of multiplicity k contributes $\binom{k}{2} < k^2/2$ to this number. Therefore, the total contribution of the points in S_i is at most

$$100\left(\frac{e^2}{2^{3i}M^3} + \frac{e}{2^iM}\right)2^{2i+1}M^2 = 100\left(\frac{e^2}{M}2^{1-i} + eM2^{i+1}\right).$$

Adding up, we obtain that the contribution of all crossings of multiplicity larger than M to the number of crossing pairs of edges is at most

$$\sum_{\substack{i \geq 0 \\ M2^i \leq n/2}} 100\left(\frac{e^2}{M}2^{1-i} + eM2^{i+1}\right) < 100\left(\frac{4e^2}{M} + 2en\right) < \frac{\text{CR}(G)}{2}.$$

Therefore, at least half of the edge crossings occur at points of multiplicity at most M, that is, at a point belonging to S. Each of these points contributes to the crossing number at most $\binom{M}{2} < \frac{M^2}{2}$. Thus, we have $|S|\frac{M^2}{2} > \frac{\text{CR}(G)}{2}$, which yields that $|S| > \frac{\text{CR}^3(G)}{(40e)^4}$. $\square$

The *bisection width*, $b(G)$, of a graph G is defined as the minimum number of edges whose removal splits the graph into two roughly equal subgraphs. More precisely, $b(G)$ is the minimum number of edges running between V_1 and V_2, over all partitions of the vertex set of G into two parts $V_1 \cup V_2$ such that $|V_1|, |V_2| \geq n(G)/3$. We need the following result.

Lemma 3.2. [5] *Let G be a graph of n vertices and e edges. Then we have*

$$b(G) \leq 10\sqrt{\mathrm{CR}(G)} + 4\sqrt{en}.$$

For the proof of Theorem 2, we pick a nested sequence of subgraphs $G = G_0 \supset G_1 \supset G_2 \supset \ldots$, according to the following procedure.

STEP 0. **Set** $G_0 := G$, $n_0 := n(G) = n$, $e_0 := e(G) = e$, and $\mathrm{CR}_0 =: \mathrm{CR}(G)$.

Suppose that we have already executed STEP i. Denote the resulting graph by G_i, let by $n_i = n(G_i)$, $e_i = e(G_i)$, $\mathrm{CR}_i = \mathrm{CR}(G_i)$, and assume that $(1/3)^i n \leq n_i \leq (2/3)^i n$.

STEP $i+1$. **If**

$$\mathrm{CR}_i \geq \left(\frac{e_i e}{n}\right)^{4/3} + 10^3 e_i n_i,$$

then STOP.

Else, delete $b(G_i)$ edges from G_i such that G_i falls into two parts, both having at most $(2/3)n_i$ vertices. Let G_i' be the resulting (disconnected) graph. Let G_{i+1} be the part in which the average degree of the vertices is at least as high as in the other.

Suppose that the algorithm terminates in STEP $I+1$.

Lemma 3.3. Suppose that $e(G) > n^{4/3}(G)$. For any $0 \leq i \leq I$ such that $e_i \geq 10^{12}(e/n)^2$, we have $\frac{e_i}{n_i} > \frac{e}{2n}$.

Proof. We prove the statement by induction on i. Obviously, it is true for $i = 0$. Let $1 \leq i \leq I$, and suppose that the lemma has been proved for all $j < i$.

Since the procedure did not stop at an earlier stage, we have

$$\mathrm{CR}_j < \left(\frac{e_j e}{n}\right)^{4/3} + 10^3 e_j n_j,$$

for every $j < i$. In view of Lemma 3.2, we obtain

$$e(G_j') = e_j - b(G_j) \geq e_j - 10\sqrt{\mathrm{CR}_j} - 4\sqrt{e_j n_j}$$

$$\geq e_j\left(1 - \frac{10(e/n)^{2/3}}{e_j^{1/3}} - 10^{3/2}\sqrt{\frac{n_j}{e_j}} - 4\sqrt{\frac{n_j}{e_j}}\right) \geq e_j\left(1 - \frac{10(e/n)^{2/3}}{e_j^{1/3}} - 40\sqrt{\frac{n_j}{e_j}}\right).$$

Using the fact that the average degree in G_{j+1} is at least as much as in G_j' and that $i \leq 2\log_2 n$, we have

$$\frac{e_i}{n_i} \geq \frac{e}{n}\prod_{0 \leq j < i}\left(1 - \frac{10(e/n)^{2/3}}{e_j^{1/3}} - 40\sqrt{\frac{n_j}{e_j}}\right)$$

$$\geq \frac{e}{n}\left(1-\sum_{0\leq j<i}\frac{10(e/n)^{2/3}}{e_j^{1/3}}-\sum_{0\leq j<i}40\sqrt{\frac{n_j}{e_j}}\right)$$

$$\geq \frac{e}{n}\left(1-10(e/n)^{2/3}\cdot 2(n/e)^{1/3}\sum_{0\leq j<i}\frac{1}{n_j^{1/3}}-80\log n\sqrt{\frac{2n}{e}}\right)$$

$$\geq \frac{e}{n}\left(1-200(e/n)^{1/3}\cdot\frac{1}{n_i^{1/3}}-80\log n\sqrt{\frac{2n}{e}}\right)>\frac{e}{2n},$$

provided that $n = n(G)$ is large enough. This concludes the proof of Lemma 3.3. □

Proof of Theorem 2. If $e \leq n^{4/3}$ then the result is an immediate consequence of Theorem 1.

Assume that $e > n^{4/3}$ and that the procedure stopped at step $I+1$. We distinguish three cases.

Case 1: Suppose that $e = e_0 < 4\cdot 10^{12}\,(e/n)^2$. Then $e > n^2/(4\cdot 10^{12})$. By the result of Ajtai, Chvátal, Newborn, Szemerédi [1], and Leighton [3], quoted in Section 1 (above Theorem 1), we have

$$\text{CR}(G) \geq \frac{1}{64}\frac{e^3}{n^2} \geq 10^{12}en,$$

if n is large enough. Therefore, we can apply Lemma 3.1 and obtain that

$$\text{CR}^*(G) \geq \frac{\text{CR}^3(G)}{(40e)^4} \geq \frac{1}{40^4}\cdot\frac{1}{64^3}\cdot\frac{e^9}{n^6e^4} = \frac{1}{40^4 64^3}\frac{e}{n^2}\cdot\frac{e^4}{n^4} > \frac{1}{10^{25}}\frac{e^4}{n^4}.$$

Case 2: Suppose that $e = e_0 \geq 4\cdot 10^{12}(e/n)^2$ and $e_I < 4\cdot 10^{12}(e/n)^2$. Clearly, for any $j < I$, $e_j \geq e_{j+1}$. Let $j < I$ be the greatest index such that $e_j \geq 4\cdot 10^{12}(e/n)^2$. Lemma 3.3 implies that $\frac{e_j}{n_j} > \frac{e}{2n} > \frac{n^{1/3}}{2}$.

We claim that $e_j \geq e_{j+1} > e_j/4$. Indeed, by definition, we have

$$e_{j+1} \geq \frac{e(G'_j)}{3} = \frac{e_j}{3}\left(1-\frac{10(e/n)^{2/3}}{e_j^{1/3}}-40\sqrt{\frac{n_j}{e_j}}\right) > \frac{e_j}{4},$$

provided that n is large enough. Hence, $10^{12}(e/n)^2 \leq e_{j+1} < 4\cdot 10^{12}(e/n)^2$. Thus, we can again apply Lemma 3.3 to obtain $\frac{e_{j+1}}{n_{j+1}} > \frac{e}{2n}$, so that $n_{j+1} < e_{j+1}\cdot(2n/e) < 4\cdot 10^{12}(e/n)^2(2n/e) = 8\cdot 10^{12}(e/n)$. The theorem of Ajtai et al. now implies that

$$\text{CR}(G_{j+1}) \geq \frac{1}{64}\frac{4^3 10^{36}}{8^2 10^{24}}\left(\frac{e}{n}\right)^4 > 10^{10}\frac{e^4}{n^4}.$$

If n is sufficiently large, we can apply Lemma 3.1 to G_{j+1} to conclude that

$$\mathrm{CR}^*(G) \geq \mathrm{CR}^*(G_{j+1}) \geq \frac{10^{30}}{40^4}\frac{(e/n)^{12}}{e_{j+1}^4} \geq \frac{10^{30}}{40^4}\frac{(e/n)^{12}}{400^4(e/n)^8} > 10^{13}\frac{e^4}{n^4}.$$

Case 3: Suppose now that $e_I \geq 4 \cdot 10^{12}(e/n)^2$. Since the procedure has stopped, we have $\mathrm{CR}_I \geq (e_I e/n)^{4/3} + 10^3 e_I n_I$. We can apply Lemma 3.1 and obtain that

$$\mathrm{CR}^*(G) \geq \mathrm{CR}^*(G_I) \geq \frac{1}{40^4}\frac{\mathrm{CR}_I^3}{e_I^4} \geq \frac{1}{40^4}\frac{e^4}{n^4}.$$

This concludes the proof of Theorem 2. $\square$

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