Crossing Families

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Abstract

Given a set of points in the plane, a crossing family is a collection of line segments, each joining two of the points, such that any two line segments intersect internally. Two sets A and B of points in the plane are mutually avoiding if no line subtended by a pair of points in A intersects the convex hull of B, and vice versa. We show that any set of n points in general position contains a pair of mutually avoiding subsets each of size at least $\sqrt{n/12}$. As a consequence we show that such a set possesses a crossing family of size at least $\sqrt{n/12}$, and describe a fast algorithm for finding such a family.

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1 Introduction

Consider n points in the plane in general position (no three points collinear). We say that a collection of line segments, each joining two of the given points, is a *crossing family* if every two segments intersect internally. In a natural variation the points belong to two color classes, and each segment of the crossing family joins points of different colors. We say that two equal-sized disjoint sets A and B can be *crossed* if there exists a crossing family exhausting A and B in which each line segment connects a point in A with a point in B.

In this paper we study crossing families. In Section 2 we show that $\Omega(\sqrt{n})$-size crossing families exist in both the colored and uncolored versions of the problem. Our proof is constructive, and yields an algorithm which can be implemented to find such a family in time $O(n \log n)$.

We obtain the result on crossing families by finding sets of points which are *mutually avoiding*. Say that a set A *avoids* a set B if no line (not line segment) subtended by a pair of points in A intersects the convex hull of B. This means that every vertex in B "sees" the points of A in the same order. The sets A and B are *mutually avoiding* if A avoids B and B avoids A. We show how to find mutually avoiding sets of size $\Omega(\sqrt{n})$. Valtr [6] has shown that this is best possible up to the constant. The result on crossing families then follows from showing that if a pair of sets A, B are mutually avoiding and of equal cardinalities then they can be crossed.

In Section 3 we characterize which pairs of sets are mutually avoiding and which can be crossed. This characterization shows that mutual avoidance is a much stronger notion than crossability, and supports our belief that the true size of a maximum crossing family grows more quickly than $\sqrt{n}$. (It could even be linear.) In Section 4 we show that the crossing family problem is equivalent to the problem of finding a collection of line segments which are pairwise "parallel:" i.e. the lines subtended by any pair of segments intersect beyond the segments.

The notions of avoidance and mutual avoidance extend naturally to higher dimensions: if A and B are sets of points in $\mathbb{R}^d$, then A avoids B if no hyperplane subtended by d points in A intersects the convex hull of B. In Section 5 we show that polynomial-sized mutually avoiding sets exist in arbitrary dimensions.

Several researchers have considered problems involving configurations of m line segments among n points in the plane. Alon and the second author [1] showed that if $m \geq 6n - 5$ then there are always three mutually disjoint line segments.

Capoyleas and the sixth author [2] showed that for $k \leq n/2$ if the points are in convex position and $m > (k-1)(2n+1-2k)$, then there is a crossing family of size k, and that this is best possible.

2 Construction of an $\Omega(\sqrt{n})$ Crossing Family

In this section we show, given n points in general position in the plane, how to find a pair of mutually avoiding sets X' and Y' of size $\Omega(\sqrt{n})$. This is achieved by finding subsets X and Y such that X avoids Y, and then subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that Y' avoids X'. Since a pair of equal-sized mutually avoiding sets can be crossed (see Corollary 1), we thus obtain a crossing family of cardinality $\min(|X'|, |Y'|)$.

We use the following well-known results:

Lemma 1. *For any line $\mathcal{L}$ in the plane and finite set of points, it is possible to find another line $\mathcal{M}$ which simultaneously splits the points in both halfplanes in any desired proportions.*

Lemma 2. [4] *Among any sequence of real numbers of length n, there is either an ascending or a descending subsequence of length $\sqrt{n}$.*

We will work in the two-color case, where X is to be chosen from among $n/2$ blue, and Y from among $n/2$ red points.

Theorem 1.
(i) Given $n/2$ red and $n/2$ blue points, there exists a crossing family of size at least $\sqrt{n/24}$.
(ii) Given n uncolored points, there exists a crossing family of size at least $\sqrt{n/12}$.

Proof: Our strategy in proving (i) has three steps.

Step 1. This is a preliminary step where the plane is partitioned by three lines (as depicted in Figure 1) so that certain regions have linearly many points of particular colors.

Specifically, first find a line $\mathcal{L}$ such that at least $n/4$ of the reds are on one side and at least $n/4$ blues on the other by moving a horizontal $\mathcal{L}$ down from $y = +\infty$ until $n/4$ of the first color, say red, are above it. Discard the blue points above $\mathcal{L}$ and the red points below it. Second, use Lemma 1 to find a line $\mathcal{M}$ such that

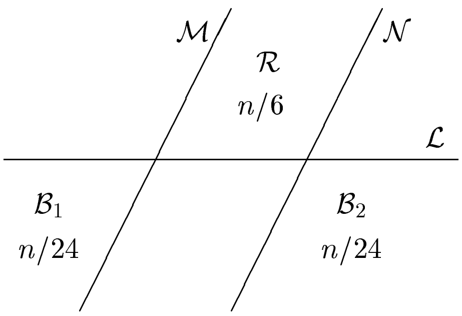

Figure 1: The H-picture

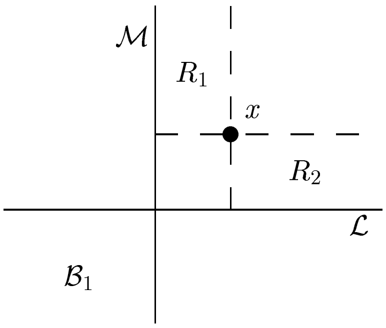

Figure 2: x splits R into two parts

exactly $n/24$ of the red and $n/24$ of the blue points are to the left of $\mathcal{M}$. Finally, take a line $\mathcal{N}$ parallel to $\mathcal{M}$ at $x = +\infty$ and move it to the left until $n/24$ of the first color, say blue, are on its right. See Figure 1. The region $\mathcal{R}$ contains at least $n/6$ red points, and the regions $\mathcal{B}_1$ and $\mathcal{B}_2$ both at least $n/24$ blue points.

Step 2. For convenience, apply an affine transformation such that $\mathcal{M}$ and $\mathcal{N}$ are vertical. Order the reds in $\mathcal{R}$ from left to right. By Lemma 2 there exists either an ascending or a descending subsequence R of length $\sqrt{n/6}$. Without loss of generality assume that R is *descending*. Then observe that R avoids $\mathcal{B}_1$.

Step 3. Consider the middle point x of R breaking it into two parts R_1 and R_2, each a descending sequence of length $\sqrt{n/24}$. See Figure 2.

Consider the positions of the blue points in $\mathcal{B}_1$ expressed in polar coordinates (r, θ) with x as the origin (and θ measured counterclockwise), and order them as $\{b_i\}$ (for $i = 1, \ldots, n/24$) in decreasing distance r_i from x. By Lemma 2 there exists a subsequence $B = \{b_{k_i}\}$ (for $i = 1, \ldots, \sqrt{n/24}$) whose angles θ_{k_i} are either decreasing or increasing. Say they are *increasing*. We claim that B avoids R_1.

For consider two points b_{k_i} and b_{k_j} of B with $i < j$. From the conditions on B it follows that b_{k_j} is to the right of b_{k_i} and below the line subtended by x and b_{k_i}. Thus the line spanned by b_{k_i} and b_{k_j} avoids the region containing R_1.

Applying Corollary 1, the theorem follows. The only change for the uncolored case (*ii*) is that $\mathcal{L}$ may be found without discarding half the points. $\square$

The above procedure provides an $O(n \log n)$-time algorithm for constructing a crossing family since one can apply Lemmas 1 and 2 in this time. (For Lemma 1 cf. [3].)

3 A Characterization

In this section we examine conditions which characterize when two sets can be crossed and when they are mutually avoiding.

Consider red points X and blue points Y separated by a line $\mathcal{L}$. We say a red point x sees a blue point y at rank i if y is the i^{th} blue point counterclockwise as seen from x. And vice versa. Then we say X and Y obey the *rank condition* if there exist labelings $x_1, \ldots, x_s$ and $y_1, \ldots, y_s$ of X and Y such that for all i, x_i sees y_i at rank i and vice versa. For the *strong rank condition*, the labelings must be such that x_i sees y_j at rank j for all i and j.

Proposition 1. *Let X and Y be s red and s blue points separated by a line. Then:*
(1) X and Y can be crossed if and only if they obey the rank condition.
(2) X and Y are mutually avoiding if and only if they obey the strong rank condition.

Since the strong rank condition implies the rank condition, this gives:

Corollary 1. *A pair of sets can be crossed if they are mutually avoiding and of equal cardinality.*

Proof: (1) Say the line $\mathcal{L}$ is vertical, with reds X on the left and blues Y on the right.

Assume first that X and Y can be crossed. Let $l_1, \ldots, l_s$ be the line segments of a complete crossing family in order of increasing slope. Label the red endpoint of l_i, x_i, and the blue endpoint y_i. Since $l_1, \ldots, l_{i-1}$ are of lesser slope than l_i, and intersect it, x_i sees $y_1, \ldots, y_{i-1}$ before it sees y_i. Similarly, x_i sees $y_{i+1}, \ldots, y_s$ after y_i and thus it sees y_i at rank i. For the same reason, y_i sees x_i at rank i.

Assume now that there exist labelings x_i, y_i satisfying the rank condition. We prove by induction on s that the family $\{x_i y_i\}_i$ of line segments is a crossing family. The case $s = 1$ is trivial.

Consider the line ℓ_s extending the segment $x_s y_s$. By the rank condition, $X - x_s$ and $Y - y_s$ lie on opposite sides of this line. Hence $x_s y_s$ intersects $x_i y_i$ if it intersects ℓ_i. Also, the slope of $x_i y_i$ is less than that of $x_s y_s$ for all $i < s$.

Let A be the set of all line segments that do not intersect $x_s y_s$; order the members of A with respect to their $\mathcal{L}$-intercepts. If A is nonempty then without loss of generality it contains a line segment whose $\mathcal{L}$-intercept is above that of $x_s y_s$; then choose a such that $x_a y_a$ has the highest $\mathcal{L}$-intercept in A. See Figure 3. Then the line ℓ_a extending $x_a y_a$ does not intersect $x_s y_s$. So there exist $a - 1$ red points above ℓ_a and $a - 1$ blue points below it. But y_s is among these blue points while x_s is not among the red ones. Thus there exists b so that x_b and y_b are both above ℓ_a. Then $x_b y_b$ does not intersect $x_s y_s$ and has a higher $\mathcal{L}$-intercept than $x_a y_a$, contrary to assumption. Hence A is empty, and $x_s y_s$ intersects all $x_i y_i$.

To prove that $x_i y_i$ intersects $x_j y_j$ for all $i < j < s$, we observe that since the slope of $x_s y_s$ is greater than that of any other segment, the rank condition is preserved upon deletion of x_s and y_s, with the same labeling $x_1, \ldots, x_{s-1}$ and $y_1, \ldots, y_{s-1}$. Hence by induction $\{x_i y_i\}_1^{s-1}$ is a crossing family.

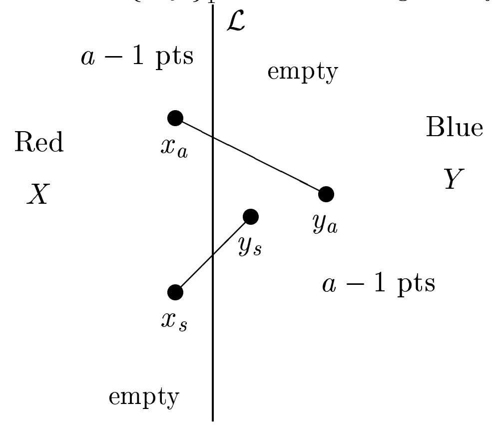

Figure 3: The picture when $x_a y_a$ is above $x_s y_s$

(2) The order in which a point x sees two points y_1 and y_2 is determined by which side of the line through them x lies. If Y avoids X then each pair y_i, y_j is seen in the same order by all $x \in X$, and a similar conclusion holds if X avoids Y; hence if X and Y are mutually avoiding then they satisfy the strong rank condition.

If on the other hand the sets are not mutually avoiding, say Y does not avoid

X, then there is a pair of points in Y subtending a line through the convex hull of X, and this pair is seen in a different order by the points of X to each side of that line. Hence the strong rank condition fails. □

4 Parallel Families

We use the term "parallel" to describe a pair of nonintersecting line segments whose extensions intersect outside both segments. We say that a family of line segments is parallel if every pair of the segments is parallel.

The following can be shown in the manner of Proposition 1:

Proposition 2. *Let X and Y be two sets, of s points each, separated by a line. Then X and Y can be paired up to form a parallel family if and only if there exist labelings $x_1, \ldots, x_s$ and $y_1, \ldots, y_s$ of X and Y such that for all i, x_i sees y_i at rank i and y_i sees x_i at rank $s+1-i$. In particular, if X and Y are mutually avoiding then X and Y can be so paired.*

The problems of finding large parallel and large crossing families are equivalent:

Theorem 2. *Let $c(n)$ (resp. $p(n)$) denote the minimum number of segments in a maximum crossing (parallel) family, among all configurations of n blue and n red points separated by a line, with all $2n$ points in general position. Then $c(n) = p(n)$.*

Proof: Consider a configuration of points with n points either side of the y-axis and the transformation f given by $(x, y) \mapsto (1/x, y/x)$. This carries the points to a new configuration such that if segments b_1r_1 and b_2r_2 intersect and meet the y-axis, then segments $f(b_1)f(r_1)$ and $f(b_2)f(r_2)$ are parallel and meet the y-axis, and vice versa. □

5 Mutually Avoiding Sets in Higher Dimensions

In this section we show that there are polynomial-sized mutually avoiding sets in arbitrary dimensions. A hyperplane *stabs* a set in $\mathbb{R}^d$ if it intersects the convex hull of that set. The *stabbing number* of a collection of sets is the maximum number of sets that any hyperplane stabs. We use the following result of Matoušek:

Lemma 3. [5] *Let P be a set of n points in $\mathbb{R}^d$ and let $r \leq n$. Then there exists a subset $P' \subseteq P$ of at least $n/2$ points and a partition $\{P_1, \ldots, P_m\}$ of P' with $|P_i| = \lfloor n/r \rfloor$ for all i and with stabbing number $O(r^{1-1/d})$.*

Theorem 3. *Any set of n points in $\mathbb{R}^d$ contains a pair of mutually avoiding subsets each of size $\Omega(n^{1/(d^2-d+1)})$.*

Proof: Say we apply the above result with parameter r yielding a partition of P' into blocks $P_1, \ldots, P_m$. Note that m is $\Theta(r)$. The points of each P_i generate $O(n^d/r^d)$ hyperplanes, and each of these hyperplanes stabs $O(r^{1-1/d})$ subsets. Thus there are at most $O(n^d r^{2-d-1/d})$ stabbings in all.

Associate every stabbing with the (unordered) pair of blocks consisting of the block generating the hyperplane, and the block which is stabbed by it. Since there are $\Theta(r^2)$ pairs of blocks, some pair has only $n^d r^{-d-1/d}$ such mutual stabbings. If r is chosen to be approximately $n^{(d^2-d)/(d^2-d+1)}$ then this pair has at most $n/2r$ mutual stabbings. Each stabbing is created by a hyperplane which can be eliminated by removing one point from one of the blocks. The depleted blocks are each of size at least $n/2r = \Omega(n^{1/(d^2-d+1)})$, and are mutually avoiding. $\square$

We can use the above result to find an analogue of a polynomial-sized crossing family in $\mathbb{R}^d$: a collection of d-simplices such that every two simplices intersect and have disjoint vertex sets. We omit the construction.

6 Discussion

We believe our lower bound on the size of maximum crossing families can be improved. Our best upper bound is linear: at most $n/2$ points used for the uncolored case (for example four non-convex points) and at most $3n/8$ in the colored case. For the latter consider the arrangement of sixteen points in Figure 4. These ratios can be obtained for arbitrarily large n by splitting points suitably. Regarding the behavior of "generic" sets of points, we note that n points (colored or not) chosen at random in the unit disk, almost surely have a linear-sized crossing family. We omit the details.

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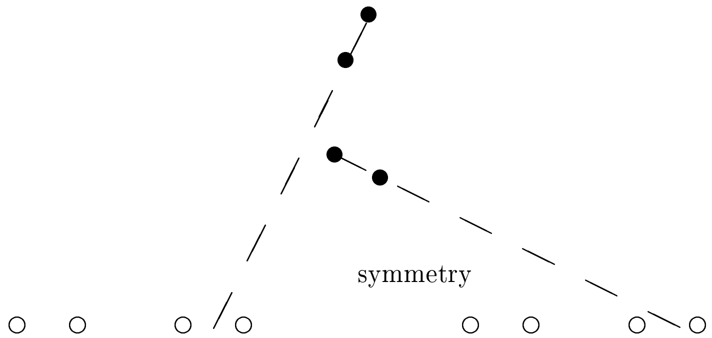

Figure 4: Arrangement for upper bound for two-colored case: 4 more black points are located symmetrically below the whites

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