A Bipartite Strengthening of the Crossing Lemma

Jacob Fox[1⋆], János Pach[2⋆⋆], and Csaba D. Tóth[3]

[1] Department of Mathematics, Princeton University, Princeton, NJ, USA
jacobfox@math.princeton.edu
[2] City College, CUNY and Courant Institute, NYU, New York, NY, USA
pach@cims.nyu.edu
[3] University of Calgary, Calgary, AB, Canada
cdtoth@ucalgary.ca

Abstract. The celebrated Crossing Lemma states that, in every drawing of a graph with n vertices and $m \geq 4n$ edges there are at least $\Omega(m^3/n^2)$ pairs of crossing edges; or equivalently, there is an edge that crosses $\Omega(m^2/n^2)$ other edges. We strengthen the Crossing Lemma for drawings in which any two edges cross in at most $O(1)$ points.

We prove for every $k \in \mathbb{N}$ that every graph G with n vertices and $m \geq 3n$ edges drawn in the plane such that any two edges intersect in at most k points has two disjoint subsets of edges, E_1 and E_2, each of size at least $c_k m^2/n^2$, such that every edge in E_1 crosses all edges in E_2, where $c_k > 0$ only depends on k. This bound is best possible up to the constant c_k for every $k \in \mathbb{N}$. We also prove that every graph G with n vertices and $m \geq 3n$ edges drawn in the plane with x-monotone edges has disjoint subsets of edges, E_1 and E_2, each of size $\Omega(m^2/(n^2 \operatorname{polylog} n))$, such that every edge in E_1 crosses all edges in E_2. On the other hand, we construct x-monotone drawings of bipartite dense graphs where the largest such subsets E_1 and E_2 have size $O(m^2/(n^2 \log(m/n)))$.

1 Introduction

The crossing number $\mathrm{cr}(G)$ of a graph[4] G is the minimum number of crossings in a drawing of G. A *drawing* of a graph G is a planar embedding which maps the vertices to distinct points in the plane and each edge to a simple continuous arc connecting the corresponding vertices but not passing through any other vertex. A *crossing* is a pair of curves and a common interior point between the two curves (the intersections at endpoints or vertices do not count as crossings). A celebrated result of Ajtai et al. [ACNS82] and Leighton [L84], known as

⋆ Research supported by an NSF Graduate Research Fellowship and a Princeton Centennial Fellowship.

⋆⋆ Supported by NSF Grant CCF-05-14079, and by grants from NSA, PSC-CUNY, Hungarian Research Foundation OTKA, and BSF.

[4] The graphs considered here are simple, having no loops or parallel edges.

the Crossing Lemma, states that the crossing number of every graph G with n vertices and $m \geq 4n$ edges satisfies

$$\mathrm{cr}(G) = \Omega\left(\frac{m^3}{n^2}\right). \tag{1}$$

The best known constant coefficient is due to [PRTT06]. Leighton [L84] was motivated by applications to VLSI design. Székely [S97] used the Crossing Lemma to give simple proofs of Szemerédi-Trotter bound on the number of point-line incidences [ST83], a bound on Erdős's unit distance problem and Erdős's distinct distance problem [E46]. The Crossing Lemma has since found many important applications, in combinatorial geometry [D98,KT04,PS98,PT02,STT02], and number theory [ENR00,TV06].

The *pairwise crossing number* pair-cr(G) of a graph G is the minimum number of pairs of crossing edges in a drawing of G. The lower bound (1) also holds for the pairwise crossing number with the same proof. It follows that in every drawing of a graph with n vertices and $m \geq 4n$ edges, there is an edge that crosses at least $\Omega(m^2/n^2)$ other edges. Conversely, if in every drawing of every graph with $m \geq 3n$ edges some edge crosses $\Omega(m^2/n^2)$ others, then we have pair-cr$(G) = \Omega(m^3/n^2)$ for every graph G with $m \geq 4n$ edges. Indeed, by successively removing edges that cross many other edges, we obtain the desired lower bound for the total number of crossing pairs. In this note, we prove a bipartite strengthening of this result for drawings where any two edges intersect in at most a constant number of points.

Theorem 1. *For every $k \in \mathbb{N}$, there is a constant $c_k > 0$ such that for every drawing of a graph $G = (V, E)$ with n vertices and $m \geq 3n$ edges, no two of which intersect in more than k points, there are disjoint subsets $E_1, E_2 \subset E$, each of size at least $c_k m^2/n^2$, such that every edge in E_1 crosses all edges in E_2.*

We have $k = 1$ in straight-line drawings, $k = (\ell + 1)^2$ if every edge is a polyline with up to ℓ *bend*s, and $k = d^2$ if the edges are sufficiently generic algebraic curves (e.g., splines) of degree at most d. Note also that every graph G has a drawing with cr(G) crossings in which any two edges cross at most once [V05].

The dependence on k in Theorem 1 is necessary: We show that one cannot expect bipartite crossing families of edges of size $\Omega(m^2/n^2)$ if any two edges may cross arbitrarily many times, even if the graph drawings are restricted to be x-monotone. An *x-monotone curve* is a continuous arc that intersects every vertical line in at most one point. A drawing of a graph is *x-monotone* if every edge is mapped to an x-monotone curve.

Theorem 2. *For every $n, m \in \mathbb{N}$ with $m \leq n^2/4$, there is a bipartite graph $G = (V, E)$ with n vertices, m edges, and an x-monotone drawing such that any two disjoint subsets $E_1, E_2 \subset E$ of equal size $|E_1| = |E_2| = t$, where every edge in E_1 crosses all edges in E_2, satisfy*

$$t = O\left(\frac{m^2}{n^2 \log(m/n)}\right).$$

We present the tools used for the bipartite strengthening of the Crossing Lemma in the next section. Theorem 1 is proved in Section 3. Our construction of x-monotone drawings are discussed in Section 4. Finally, Section 5 contains a weaker analogue of Theorem 1 for x-monotone drawings and a further strengthening of the Crossing Lemma for graphs satisfying some monotone property.

2 Tools

The proof of Theorem 1 relies on a recent result on the intersection pattern of k-intersecting curves. For a collection C of curves in the plane, the *intersection graph* is defined on the vertex set C, two elements of C are *adjacent* if the (relative) interiors of the corresponding curves intersect. A complete bipartite graph is *balanced* if the vertex classes differ in size by at most one. For brevity, we call a balanced complete bipartite graph a *bi-clique*.

Theorem 3. [FPT07a] *Given m curves in the plane such that at least εm^2 pairs intersect and any two curves intersect in at most k points, their intersection graph contains a bi-clique with at least $c_k \varepsilon^{64} m$ vertices where $c_k > 0$ depends only on k.*

If follows from the Crossing Lemma that in every drawing of a dense graph, the intersection graph of the edges is also dense. Therefore, Theorem 3 implies Theorem 1 in the special case that G is dense. This connection was first observed by Pach and Solymosi [PS01] who proved Theorem 1 for straight-line drawings of dense graphs.

If a graph G is *not* dense, we decompose G recursively into induced subgraphs with an algorithm reminiscent of [PST00] until one of the components is dense enough so that Theorem 3, like before, implies Theorem 1. The decomposition algorithm successively removes *bisectors*, and we use Theorem 4 below to keep the total number of deleted edges under control.

The *bisection width*, denoted by $b(G)$, is defined for every simple graph G with at least two vertices. It is the smallest nonnegative integer such that there is a partition of the vertex set $V = V_1 \cup^* V_2$ with $\frac{1}{3} \cdot |V| \leq V_i \leq \frac{2}{3} \cdot |V|$ for $i = 1, 2$, and $|E(V_1, V_2)| = b(G)$. Pach, Shahrokhi, and Szegedy [PSS96] gave an upper bound on the bisection width in terms of the crossing number and the L_2-norm of the degree vector (it is an easy consequence of the weighted version of the famous Lipton-Tarjan separator theorem [LT79,GM90]).

Theorem 4. [PSS96] *Let G be a graph with n vertices of degree $d_1, d_2, \ldots, d_n$. Then*

$$b(G) \leq 10\sqrt{\mathrm{cr}(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2(G)}. \tag{2}$$

3 Proof of Theorem 1

Let $G = (V, E)$ be a graph with n vertices and $m \geq 3n$ edges. Since a graph with more than $3n-6$ edges cannot be planar, it must have crossing edges. Hence, as long as $3n \leq m < 10^6 n$, Theorem 1 holds with $|E_1| = |E_2| = 1 \geq 10^{-12} m^2/n^2$. We assume $m \geq 10^6 n$ in the remainder of the proof.

Let D be a drawing of G. To use the full strength of Theorem 4, we transform the drawing D into a drawing D' of a graph $G' = (V', E')$ with m edges, at most $2n$ vertices, and maximum degree at most $\lceil 2m/n \rceil$, so that the intersection graph of E' is isomorphic to that of E. If the degree of a vertex $v \in V$ is above the average degree $\bar{d} = 2m/n$, split v into $\lceil d/\bar{d} \rceil$ vertices $v_1, \ldots, v_{\lceil d/\bar{d} \rceil}$ arranged along a circle of small radius centered at v. Denote the edges of G incident to v by $(v, w_1), \ldots, (v, w_d)$ in clockwise order in the drawing D. In G', connect w_j with v_i if and only if $\bar{d}(i-1) < j \leq \bar{d}i$, where $1 \leq j \leq d$ and $1 \leq i \leq \lceil d/\bar{d} \rceil$. Two edges of G' cross if and only if the corresponding edges of G cross. Also, letting $d(v)$ denote the degree of vertex v in G', the number of vertices of G' is

$$\sum_{v \in V} \lceil d(v)/\bar{d} \rceil < \sum_{v \in V} 1 + d(v)/\bar{d} = 2n.$$

Hence the resulting G' and D' have all the required properties.

We will decompose G' recursively into induced subgraphs until each induced subgraph is either a singleton or it has so many pairs of crossing edges that Theorem 3 already implies Theorem 1. Theorem 3 implies that the intersection graph of the edges of an induced subgraph H of G' contains a bi-clique of size at least $c_k \left(\frac{p(H)}{e(H)^2}\right)^{64} e(H)$, where $p(H)$ is the number of pairs of crossing edges in H in the drawing D', $e(H)$ is the number of edges of H, and $c_k > 0$ is the constant depending on k only in Theorem 3. So the intersection graph of the edge set of G' (and hence also of G) contains a bi-clique of size $\Omega_k(m^2/n^2)$ if there is an induced subgraph H of G' with

$$\varepsilon_k \frac{m^2}{n^2} \leq \left(\frac{p(H)}{e(H)^2}\right)^{64} e(H), \tag{3}$$

where $\varepsilon_k > 0$ is any constant depending on k only. We use $\varepsilon_k = (10^9 k)^{-64}$ for convenience. Assume, to the contrary, that (3) does not hold for any induced subgraph H of G'.

Every induced subgraph H has at most $kp(H)$ crossings in the drawing D', hence $\mathrm{cr}(H) \leq kp(H)$. It is enough to find an induced subgraph H for which

$$\frac{e(H)^{2-1/64}}{10^9} \left(\frac{m}{n}\right)^{\frac{1}{32}} \leq \mathrm{cr}(H), \tag{4}$$

since this combined with $\mathrm{cr}(H) \leq kp(H)$ implies (3).

Next, we decompose the graph G' of at most $2n$ vertices and m edges with the following algorithm.

Decomposition Algorithm

1. Let $S_0 = \{G'\}$ and $i = 0$.
2. While $(3/2)^i \le 4n^2/m$ and no $H \in S_i$ that satisfies (4), do
   Set $i := i + 1$. Let $S_i := \emptyset$. For every $H \in S_{i-1}$, do
   - If $|V(H)| \le (2/3)^i\, 2n$, then let $S_i := S_i \cup \{H\}$;
   - otherwise split H into induced subgraphs H_1 and H_2 along a bisector of size $b(H)$, and let $S_i := S_i \cup \{H_1, H_2\}$.
3. Return S_i.

For every i, every graph $H \in S_i$ satisfying the end condition has at most $|V(H)| \le (2/3)^i\, 2n$ vertices. Hence, the algorithm terminates in $t \le \log_{(3/2)} 2n$ rounds and it returns a set S_t of induced subgraphs. Let $T_i \subset S_i$ be the set of those graphs in S_i that have more than $(2/3)^i\, 2n$ vertices. Notice that $|T_i| \le (3/2)^i$. Denote by G_i the disjoint union of the induced subgraphs in S_i.

We use Theorem 4 for estimating the number of edges deleted throughout the decomposition algorithm. Substituting the upper bound for $\mathrm{cr}(H)$ and using Jensen's inequality for the concave function $f(x) = x^{1-1/128}$, we have for every $i = 0, 1, \ldots, t$,

$$\sum_{H\in T_i} \sqrt{\mathrm{cr}(H)} \le \sum_{H\in T_i} \sqrt{\frac{e(H)^{2-1/64}}{10^9}\left(\frac{m}{n}\right)^{\frac{1}{32}}} = 10^{-\frac{9}{2}}\left(\frac{m}{n}\right)^{\frac{1}{64}} \sum_{H\in T_i} e(H)^{1-\frac{1}{128}}$$
$$\le 10^{-\frac{9}{2}}\left(\frac{m}{n}\right)^{\frac{1}{64}} |T_i|^{\frac{1}{128}} m^{1-\frac{1}{128}} \le 10^{-\frac{9}{2}}\left(\frac{3}{2}\right)^{\frac{i}{128}} \frac{m^{1+1/128}}{n^{1/64}}.$$

Denoting by $d(v, H)$ the degree of vertex v in an induced subgraph H, we have

$$\sum_{H\in T_i} \sqrt{\sum_{v\in V(H)} d^2(v, H)} \le \sqrt{|T_i|}\sqrt{\sum_{v\in V(G_i)} d^2(v, G_i)}$$
$$\le \sqrt{(3/2)^i}\sqrt{n\cdot(\bar{d})^2} \le \frac{2m}{\sqrt{n}}\sqrt{(3/2)^i}.$$

In the first of the two above inequalities, we use the Cauchy-Schwartz inequality to get $\sum_{H\in T_i}\sqrt{x_H} \le \sqrt{|T_i|}\sqrt{\sum_{H\in T_i} x_H}$ with $x_H = \sum_{v\in V(H)} d^2(v, H)$.

By Theorem 4, the total number of edges deleted during this process is

$$\sum_{i=0}^{t-1}\sum_{H\in T_i} b(H) \le 10\sum_{i=0}^{t-1}\sum_{H\in T_i}\sqrt{\mathrm{cr}(H)} + 2\sum_{i=0}^{t-1}\sum_{H\in T_i}\sqrt{\sum_{v\in V(H)} d^2(v, H)}$$
$$\le 10^{-\frac{7}{2}}\frac{m^{1+1/128}}{n^{1/64}}\sum_{i=0}^{t-1}(3/2)^{\frac{i}{128}} + 4\frac{m}{\sqrt{n}}\sum_{i=0}^{t-1}\sqrt{(3/2)^i}$$
$$\le \frac{m^{1+1/128}}{4n^{1/64}}\left(\frac{n^2}{m}\right)^{1/128} + 100 m^{1/2} n^{1/2} \le \frac{m}{2}.$$

The second inequality uses the earlier upper bounds for $\sum_{H \in T_i} \sqrt{\mathrm{cr}(H)}$ and $\sum_{H \in T_i} \sqrt{\sum_{v \in V(H)} d^2(v, H)}$, the third inequality uses the geometric series formula and the upper bound $t \le \log_{(3/2)} 2n$, while the last inequality follows from the fact that $m \ge 10^6 n$.

So at least $m/2$ edges survive and each of the induced subgraphs in S_t has at most $(2/3)^t\, 2n \le 2n/(4n^2/m) = m/2n$ vertices. Also G' has at most $2n$ vertices, so using Jensen's inequality for the convex function $g(x) = \binom{x}{2}$, the total number of vertex pairs lying in a same induced subgraph of S_t is less than

$$\frac{2n}{m/2n} \frac{(m/2n)^2}{2} = \frac{m}{2},$$

a contradiction. We conclude that the decomposition algorithm must have found an induced subgraph H satisfying (4). This completes the proof of Theorem 1. $\square$

4 Drawings with Edges as x-monotone Curves

It is known that Theorem 3 does not hold without the assumption that any two curves intersect in at most a constant number of points. Using a construction from [F06], Pach and G. Tóth [PT06] constructed for every $n \in \mathbb{N}$, a collection of n x-monotone curves whose intersection graph is dense but every bi-clique it contains has at most $O(n/\log n)$ vertices. Theorem 2 shows a stronger construction holds: the curves are edges in an x-monotone drawing of a dense bipartite graph, where $\Theta(n^2)$ curves have only n distinct endpoints.

The proof of Theorem 3 builds on a crucial observation: Golumbic et al. [GRU83] noticed a close connection between intersection graphs of x-monotone curves and partially ordered sets. Consider n continuous functions $f_i : [0,1] \to \mathbb{R}$. The graph of every continuous real function is clearly an x-monotone curve. Define the partial order $\prec$ on the set of functions by $f_i \prec f_j$ if and only if $f_i(x) < f_j(x)$ for all $x \in [0,1]$. Two x-monotone curves intersect if and only if they are incomparable under this partial order $\prec$.

Lemma 1. [GRU83] *The elements of any partially ordered set* $(\{1,2,\ldots,n\}, \prec)$ *can be represented by continuous real functions* $f_1, f_2, \ldots, f_n$ *defined on the interval* $[0,1]$ *such that* $f_i(x) < f_j(x)$ *for every* x *if and only if* $i \prec j$ $(i \neq j)$*.*

Proof. Let $(\{1,2,\ldots,n\}, \prec)$ be a partial order, and let Π denote the set consisting of all of its extensions $\pi(1) \prec \pi(2) \prec \ldots \prec \pi(n)$ to a total order. Clearly, every element of Π is a permutation of the numbers $1, 2, \ldots, n$. Let $\pi_1, \pi_2, \ldots, \pi_t$ be an arbitrary labeling of the elements of Π. Assign distinct points $x_k \in [0,1]$ to each π_k such that $0 = x_1 < x_2 < \ldots < x_t = 1$. For each i $(1 \le i \le n)$, define a continuous, piecewise linear function $f_i(x)$, as follows. For any k $(1 \le k \le t)$, set $f_i(x_k) = \pi_k^{-1}(i)$, and let $f_i(x)$ be linear over each interval $[x_k, x_{k+1}]$.

Obviously, whenever $i \prec j$ for some $i \neq j$, we have that $\pi_k^{-1}(i) \prec \pi_k^{-1}(j)$ for every k, and hence $f_i(x) < f_j(x)$ for all $x \in [0,1]$. On the other hand, if i

and j are incomparable under the partial order $\prec$, there are indices k and k' ($1 \leq k \neq k' \leq m$) such that $f_i(x_k) < f_j(x_k)$ and $f_i(x_k) > f_j(x_{k'})$, therefore, by continuity, the graphs of f_i and f_j must cross at least once in the interval $(x_k, x_{k'})$. This completes the proof. □

The following lemma is the key for the proof of Theorem 2. It presents a partially ordered set of size n^2 whose incomparability graph contains bi-cliques of size at most $O(n^2/\log n)$, yet it can be represented with x-monotone curves having only $2n$ endpoints.

Lemma 2. *For every $n \in \mathbb{N}$, there is a partially ordered set P with n^2 elements satisfying the following properties*

1. *every bi-clique in the incomparability graph of P has size at most $O(n^2/\log n)$,*
2. *there are equitable partitions $P = P_1 \cup \ldots \cup P_n$ and $P = Q_1 \cup \ldots \cup Q_n$ such that*
   (a) *for each i, there is a linear extension of P where the elements of P_i are consecutive,*
   (b) *there is a linear extension of P where the elements of each Q_j are consecutive, and*
   (c) *for every i and j, we have $|P_i \cap Q_j| = 1$.*

We now prove Theorem 2, pending the proof of Lemma 2. Note that it suffices to prove Theorem 2 in the case $m = n^2/4$, that is, when G is a bi-clique. By deleting some of the edges of this construction, we obtain a construction for every $m \leq n^2/4$, since edge deletions also decrease the intersection graph of the edges. So it is enough to prove the following.

Lemma 3. *There is an x-monotone drawing of $K_{n,n}$ such that every bi-clique in the intersection graph of the edges has size at most $O(n^2/\log n)$.*

Proof. Let P be a poset described in Lemma 2. Represent P with x-monotone curves as in the proof of Lemma 1 such that the last linear extension π_t has property (b) of Lemma 2, that is, the elements of each Q_j are consecutive in π_t.

We transform the n^2 x-monotone curves representing P into an x-monotone drawing of $K_{n,n}$. We introduce two vertex classes, each of size n, as follows. Along the line $x = 1$, the right endpoints of the x-monotone curves in each Q_j are consecutive. Introduce a vertex on $x = 1$ for each Q_j, and make it the common right endpoint of all curves in Q_j by deforming the curves over the interval $(x_{t-1}, 1]$ but keeping their intersection graph intact. These n vertices along the line $x = 1$ form one vertex class of $K_{n,n}$.

For each i, there is a vertical line $x = x_i$ along which the x-monotone curves in P_i are consecutive. Introduce a vertex for each P_i on line $x = x_i$, and make it the common left endpoint of all curves in P_i by deforming the curves over the interval $[x_i, x_{i+1})$ and erasing their portion over the interval $[0, x_i)$. These n vertices, each lying on a line $x = x_i$, form the second vertex class of $K_{n,n}$. After truncating and slightly deforming the n^2 curves representing P, we have constructed an x-monotone drawing of $K_{n,n}$.

Note that the intersection graph of the edges of this drawing of $K_{n,n}$ is a subgraph of the incomparability graph of P, so every bi-clique of the intersection graph of the edges has size at most $O(n^2/\log n)$. □

Proof of Lemma 2. We start out with introducing some notation for directed graphs. For a subset S of vertices in a directed graph G, let $N_+(S)$ denote the set of vertices x in G such that there is a vertex $s \in S$ with an edge (s, x) in G. Similarly, $N_-(S)$ is the set of vertices y in G such that there is a vertex $s \in S$ with an edge (y, s) in G. A directed graph has *path-girth* k if k is the smallest positive integer for which there are vertices x and y having at least two distinct walks of length k from x to y. Equivalently, denoting the adjacency matrix of G by A_G, it has path-girth k if $A_G^1, \ldots, A_G^{k-1}$ are all 0-1 matrices, but the matrix A_G^k has an entry greater than 1.

A *directed graph* $H = (X, E)$ is an ϵ-expander if both $N_+(S)$ and $N_-(S)$ has size at least $(1+\epsilon)|S|$ for all $S \subset V$ with $1 \leq |S| \leq |V|/2$. An *expander* is a directed graph with constant expansion.

We will use that for every $v \in \mathbb{N}$, there is a constant degree expander with v vertices and path-girth $\Omega(\log v)$. This can be proved by a slight alteration of a random constant degree directed graph. We suppose for the remainder of the proof that $H = (X, E)$ is an ϵ-expander with v vertices, maximal degree at most d, and path-girth greater than $c \log v$, where ϵ, c, and d are fixed positive constants.

For every $a \in \mathbb{N}$, we define a poset $P(a, H)$ with ground set $X \times \{1, 2, \ldots, a\}$, generated by the relations $(j_1, k_1) \prec (j_2, k_2)$ whenever $k_2 = k_1 + 1$ and (j_1, j_2) is an edge of H.

Let $P_0 = P(a, H)$ with $a = \lfloor \min\left(c, (10 \log d)^{-1}\right) \cdot \log v \rfloor$. One can show, by essentially the same argument as in [F06], that the partially ordered set P_0 has the following three properties.

1. P_0 has $a|X| = \Theta(v \log v)$ elements,
2. each element of P_0 is comparable with fewer than $d^a \leq v^{1/10}$ other elements of P_0, and
3. the largest bi-clique in the incomparability graph of P_0 has size at most $O(|X|) = O(v)$.

Since the path-girth of H is greater than a, if $x, y, z, w \in P_0$ satisfy both $w \prec y \prec x$ and $w \prec z \prec x$, then y and z must be comparable. That is, the poset in Figure 1(a) cannot be a subposet of P_0. The poset P required for Lemma 2 will be a linear size subposet of P_0. We next describe the construction of P.

A *chain* is a set of pairwise comparable elements. The maximum chains in P_0 each have size a, having one element from each of $X \times \{i\}$, $i = 1, 2, \ldots, a$. Greedily choose as many disjoint chains of size a as possible from P_0, denote the set of chains by $\mathcal{C} = \{C_1, \ldots, C_w\}$, where w is the number of chains. By the expansion property of H, we have $w = \Theta(|X|) = \Theta(v)$.

We choose greedily disjoint subsets $P_1, \ldots, P_{ha}$ of P, each of which is the union of $h = \Theta\left(\sqrt{v}\right)$ chains of $\mathcal{C}$. Each P_i has the property that, besides the comparable pairs within each of the the h chains, there are no other comparable

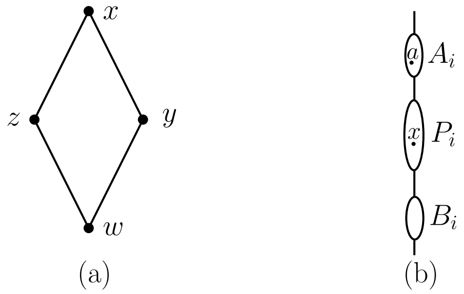

Fig. 1. (a) The Hasse diagram of a four element excluded subposet of P_0. (b) A linear extension of P where $B_i \prec P_i \prec A_i$.

pairs in P_i. We can choose the h chains of each P_i greedily: after choosing the k^{th} chain in P_i, we have to choose the $(k+1)^{\text{th}}$ chain such that none of its elements are comparable with any element of the first k chains of P_i. Since at most $kav^{1/10} \leq hav^{1/10} = v^{3/5+o(1)}$ of the $w-(i-1)h-k = \Theta(v)$ remaining chains contain an element comparable with the first k chains of P_i, almost any of the remaining chains can be chosen as the $(k+1)^{\text{th}}$ chain of P_i. Finally, let $P = P_1 \cup \ldots \cup P_{ha}$. As mentioned earlier, we have $|P| = \Theta(|P_0|)$, and the largest bi-clique in the incomparability graph of P is of size $O(|P_0|/\log|P_0|) = O(|P|/\log|P|)$.

Since the poset in Fig. 1(a) is not a subposet of P_0, no element of $P_0 \setminus C_k$, $C_k \in \mathcal{C}$, can be both greater than an element of C_k and less than another element of C_k. By construction, if two elements of P_i are comparable, then they belong to the same chain. Therefore, no element of $P \setminus P_i$ can be both greater than an element of P_i and less than another element of P_i.

Consider the partition $P = A_i \cup P_i \cup B_i$, where an element $a \in P \setminus P_i$ is in A_i if and only if there is an element $x \in P_i$ such that $x \prec a$. There is a linear extension of P in which the elements of A_i are the largest, followed by the elements of P_i, and the elements of B_i are the smallest (see Fig. 1(b)). This is because no element of $P \setminus P_i$ can be both greater than an element of P_i and less than another element of P_i.

Partition P into subsets $P = X_1 \cup \ldots \cup X_a$, where X_j consists of the elements $(j,x) \in P$ with $x \in X$. Each X_j contains exactly h^2a elements, h elements from each P_i. Arbitrarily partition each X_j into h sets $X_j = Q_{(j-1)h+1} \cup \ldots \cup Q_{jh}$ such that each Q_k contains one element from each P_i. Since the elements in each X_j form an *antichain* (a set of pairwise incomparable elements), any linear order of the elements of P for which the elements of X_j are smaller than the elements of X_k for $1 \leq j < k \leq a$ is a linear extension of P. Hence, there is a linear extension of P such that, for each j, the elements of every Q_j are consecutive.

We have established that P has all the desired properties. We can choose v such that $n \leq ha$ and $ha = O(n)$, so $v = \Theta(n^2/\log n)$. If ha is not exactly n, we may simply take the subposet whose elements are $(P_1 \cup \ldots \cup P_n) \cap (Q_1 \cup \ldots \cup Q_n)$. This completes the proof of Lemma 2. $\square$

5 Concluding Remarks

We can prove a weaker form of Theorem 1 for x-monotone curves, since our main tools (Theorems 3 and 4) are available in weaker forms in this case. It was recently shown in [FPT07b] that there is a constant $c > 0$ such that the intersection graph G of any n x-monotone curves, at least εn^2 pairs of which intersect, contains a bi-clique with at least $c\varepsilon^2 n/(\log \frac{1}{\varepsilon} \log n)$ vertices. The Crossing Lemma implies that the intersection graph of the edges of a dense topological graph is dense, so we have the following corollary.

Corollary 1. *For every x-monotone drawing of a graph $G = (V, E)$ with n vertices and $m = \Omega(n^2)$ edges, there are disjoint subsets $E_1, E_2 \subset E$, each of size at least $\Omega(n^2/\log n)$, such that every edge in E_1 crosses all edges in E_2.*

Corollary 1 is tight up to a constant factor by Theorem 2. Similar to Theorem 4, Kolman and Matoušek [KM04] proved an upper bound on the bisection width in terms of the *pairwise* crossing number and the L_2 norm of the degree sequence $d_1, d_2, \ldots, d_n$:

$$b(G) = O\left(\left(\sqrt{\text{pair-cr}(G)} + \sqrt{\sum_{i=1}^{n} d_i^2(G)}\right) \log n\right).$$

Using the same strategy as in the proof of Theorem 1, with the above mentioned tools instead of Theorems 3 and 4, it is straightforward to establish the following.

Theorem 5. *For every x-monotone drawing of a graph $G = (V, E)$ with n vertices and $m \geq 3n$ edges, there are disjoint subsets $E_1, E_2 \subset E$, each of cardinality at least $m^2/(n^2 \log^{5+o(1)} n)$, such that every edge in E_1 crosses every edge in E_2.*

In a special case, we can prove the same bound as in Theorem 1.

Proposition 1. *Given a bipartite graph G with n vertices and $m \geq 3n$ edges, and an x-monotone drawing where the vertices of the two vertex classes lie on the lines $x = 0$ and $x = 1$, respectively, then the intersection graph of the edges contains a bi-clique of size $\Omega(m^2/n^2)$.*

Proof. Consider the two dimensional partial order $\prec$ on the edges of G, where an edge e_1 is greater than another edge e_2 if and only if, for $j = 0, 1$ the endpoint of e_1 on the line $x = j$ lies above that of e_2. Two edges of G must cross if they are incomparable by the partial order $\prec$. Also notice that there is an x-monotone drawing of G with the vertices in the same position where two edges of G cross if and only if they are incomparable under $\prec$. Indeed, this is done by drawing the edges as straight line segments.

By the Crossing Lemma, there are at least $\Omega(m^3/n^2)$ pairs of crossing edges in this straight-line drawing of G. Hence, there are at least $\Omega(m^3/n^2)$ pairs of incomparable elements under the partial order $\prec$. In [FPT07b] (Theorem 3), we prove that any incomparability graph with m vertices and at least dm edges contains a bi-clique of size at least d, so the intersection graph of the edges of G must contain a bi-clique of size $\Omega(m^2/n^2)$. □

Proposition 1 implies that Theorem 1 holds for x-monotone drawings if the vertex set lies in a bounded number of vertical lines. Indeed, an x-monotone drawing of a graph with all vertices contained in the union of d vertical lines can be partitioned into $\binom{d}{2}$ x-monotone drawings of bipartite graphs with each vertex class lying on a vertical line.

Monotone properties. If a graph is drawn with at most k crossings between any two edges and the graph has some additional property, then one may improve on the bound of Theorem 1.

A graph property $\mathcal{P}$ is *monotone* if whenever a graph G satisfies $\mathcal{P}$, every subgraph of G also satisfies $\mathcal{P}$, and whenever graphs G_1 and G_2 satisfy $\mathcal{P}$, then their disjoint union also satisfies $\mathcal{P}$. The *extremal number* $\mathrm{ex}(n, \mathcal{P})$ denotes the maximum number of edges that a graph with property $\mathcal{P}$ on n vertices can have. For graphs satisfying a monotone graph property, the bound (1) of the Crossing Lemma can be improved [PST00]. In particular, if $\mathcal{P}$ is a monotone graph property and $\mathrm{ex}(n, \mathcal{P}) = O(n^{1+\alpha})$ for some $\alpha > 0$, then there exist constants $c, c' > 0$ such that for every graph G with n vertices, $m \geq cn \log^2 n$ edges, and property $\mathcal{P}$, the crossing number is at least $\mathrm{cr}(G) \geq c' m^{2+1/\alpha}/n^{1+1/\alpha}$. Furthermore, if $\mathrm{ex}(n, \mathcal{P}) = \Theta(n^{1+\alpha})$, then this bound is tight up to a constant factor. A straightforward calculation shows, using the same strategy as in the previous section, the following strengthening of Theorem 1.

Theorem 6. *Let $\mathcal{P}$ be a monotone graph property such that $ex(n, \mathcal{P}) = O(n^{1+\alpha})$ for some $\alpha > 0$. For every $k \in \mathbb{N}$, there exist positive constants c and c_k such that for any drawing of a graph $G = (V, E)$ satisfying property $\mathcal{P}$, having n vertices and $m \geq cn \log^2 n$ edges, no two of which intersecting in more than k points, there are disjoint subsets $E_1, E_2 \subset E$, each of cardinality at least $c_k(m/n)^{1+1/\alpha}$, such that every edge in E_1 crosses all edges in E_2.*

Acknowledgment. We thank the anonymous referee for bringing a paper by Golumbic, Rotem, and Urrutia [GRU83] to our attention.

References

[ACNS82] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi: Crossing-free subgraphs, in *Theory and Practice of Combinatorics,* vol. 60 of Mathematical Studies, North-Holland, Amsterdam, 1982, pp. 9–12.

[D98] T. K. Dey: Improved bounds for planar k-sets and related problems. *Discrete Comput. Geom.* **19** (1998), 373–382.

[ENR00] G. Elekes, M. B. Nathanson, and I. Z. Ruzsa: Convexity and sumsets. *J. Number Theory* **83** (2000), 194–201.

[E46] P. Erdős: On sets of distances of n points, *Amer. Math. Monthly* **53** (1946), 248–250.

[F06] J. Fox: A bipartite analogue of Dilworth's theorem, *Order* **23** (2006), 197–209.

[FPT07a] J. Fox, J. Pach, and C. D. Tóth: Intersection patterns of curves, manuscript, 2007. Cf. `http://math.nyu.edu/ pach/publications/justcurves050907.pdf`

[FPT07b] J. Fox, J. Pach, and C. D. Tóth: Turán-type results for partial orders and intersection graphs of convex sets, *Israel J. Math.* (2007), to appear.

[GRU83] M. C. Golumbic, D. Rotem, and J. Urrutia, Comparability graphs and intersection graphs. *Discrete Math.* **43** (1983), 37–46.

[GM90] H. Gazit and G. L. Miller: Planar separators and the Euclidean norm, *SIGAL Internat. Sympos. on Algorithms*, vol. 450 of LNCS, Springer, 1990, pp. 338–347.

[KT04] N. H. Katz and G. Tardos: A new entropy inequality for the Erdős distance problem, in *Towards a theory of geometric graphs*, vol. 342 of Contemp. Math., AMS, 2004, pp. 119–126.

[KM04] P. Kolman and J. Matoušek: Crossing number, pair-crossing number, and expansion, *J. Combin. Theory Ser. B* **92** (2004), 99–113.

[L84] T. Leighton: New lower bound techniques for VLSI, *Math. Systems Theory* **17** (1984), 47–70.

[LT79] R. J. Lipton and R. E. Tarjan: A separator theorem for planar graphs, *SIAM J. Appl. Math.* **36** (1979), 177–189.

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak: Ramanujan graphs, *Combinatorica* **8** (1988), 261–277.

[PRTT06] J. Pach, R. Radoičić, G. Tardos, and G. Tóth: Improving the Crossing Lemma by finding more crossings in sparse graphs, *Discrete Comput. Geom.* **36** (2006), 527-552.

[PSS96] J. Pach, F. Shahrokhi, and M. Szegedy: Applications of the crossing number, *Algorithmica* **16** (1996), 111–117.

[PS98] J. Pach and M. Sharir: On the number of incidences between points and curves. *Combin. Probab. Comput.* **7** (1) (1998), 121–127.

[PS01] J. Pach and J. Solymosi: Crossing patterns of segments, *J. Combin. Theory Ser. A* **96** (2001), 316–325.

[PST00] J. Pach, J. Spencer, and G. Tóth: New bounds on crossing numbers, *Discrete Comput. Geom.* **24** (4) (2000), 623–644.

[PT00] J. Pach and G. Tóth: Which crossing number is it anyway? *J. Combin. Theory Ser. B* **80** (2000), 225–246.

[PT02] J. Pach and G. Tardos: Isosceles triangles determined by a planar point set. *Graphs Combin.* **18** (2002), 769–779.

[PT06] J. Pach, G. Tóth: Comment on Fox News, *Geombinatorics* **15** (2006), 150–154.

[ST01] J. Solymosi, C. D. Tóth: Distinct distances in the plane. *Discrete Comput. Geom.* **25** (2001), 629–634.

[STT02] J. Solymosi, G. Tardos, and C. D. Tóth: The k most frequent distances in the plane. *Discrete Comput. Geom.* **28** (2002), 639–648.

[S97] L. A. Székely: Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.* **6** (1997), 353–358.

[ST83] E. Szemerédi and W. T. Trotter: Extremal problems in discrete geometry. *Combinatorica* **3** (1983), 381–392.

[TV06] T. Tao and V. Vu: *Additive combinatorics.* Cambridge Studies in Advanced Mathematics, 105. Cambridge University Press, Cambridge, 2006.

[V05] P. Valtr, On the pair-crossing number, in *Combinatorial and computational geometry*, vol. 52 of MSRI Publications, Cambridge Univ. Press, 2005, pp. 569–575.