On the independence number of coin graphs

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Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ be a system of non-overlapping unit disks (coins) in the plane. A subset $\mathcal{C}' \subseteq \mathcal{C}$ is called *independent* if no two members of $\mathcal{C}'$ touch each other. In 1983, P. Erdős raised the following question.

Problem. *What is the largest number $F = F(n)$ with the property that every system of n non-overlapping unit disks in the plane has an independent subsystem with at least F members?*

Associate a graph *(coin graph)* with $\mathcal{C}$ by assigning to each disk C_i its center c_i, $1 \leq i \leq n$, and connecting two points c_i and c_j with an edge if and only if the corresponding disks touch each other. As was pointed out by R. Pollack [P85], this graph is planar, and hence 4-colorable. As each color class forms an independent set, and the largest color class has at least $n/4$ members, we obtain $F(n) \geq \lceil n/4 \rceil$. (The 4-colorability of such a coin graph can be proved by a simple induction argument, without referring to the Four Color Theorem.) Recently, Gy. Csizmadia [C96] managed to improve the above bound to

$$F(n) \geq \left\lceil \frac{9}{35} n \right\rceil > 0.257n.$$

On the other hand, the 19-point graph of F. Chung, R. L. Graham,

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and J. Pach (see [E87,PA95]) depicted in Figure 1 illustrates that

$$F(n) \leq \left\lceil \frac{6}{19} n \right\rceil \approx 0.316n.$$

In this graph, every independent set S has at most 6 points. Indeed, we can assume by symmetry that $q \notin S$, and each shaded triangle contains at most one element of S.

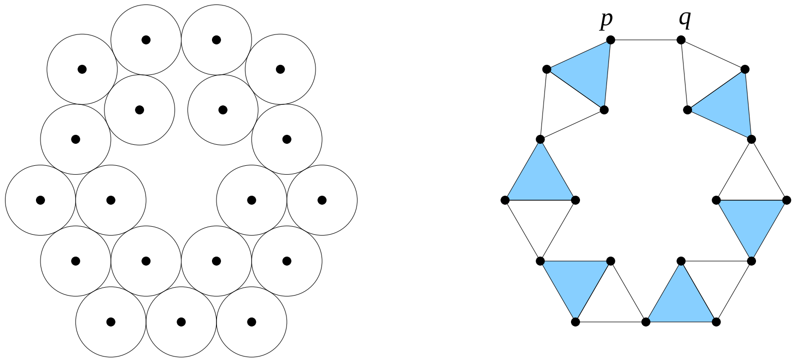

Figure 1. Nineteen coins and the corresponding coin graph

For every n, one can build a coin graph of n vertices, with $\lfloor n/19 \rfloor$ components isomorphic to the graph shown in Figure 1. Let the remaining vertices induce as many independent triangles as possible, and an extra edge or vertex, if necessary. The size of the largest independent set in this graph is $\lceil 6n/19 \rceil$.

Here we present a better construction, which yields the following.

Theorem. *$F(n) \leq \lceil \frac{5}{16} n \rceil \approx 0.3125n$ for every sufficiently large n.*

Proof: Consider the graph B_1 of 16 vertices, the first block depicted in Figure 2. Every independent set of vertices in B_1 has at most 6 elements: at most one point from each shaded triangle and possibly r_1. For every sufficiently large k, one can build a closed chain C by putting together k congruent copies of B_1 in the way indicated in the figure, so that the orientation of B_{i+1} is either the same as or opposite to the orientation of B_i, $i = 1, 2, \ldots, k-1$. Notice that this figure is not rigid. We can slightly deform the picture without violating the conditions that (a) no two vertices are closer than one, (b) the distance between any two adjacent vertices is exactly one.

Every block $B_i \subset C$ has at most 6 independent vertices. We claim that in average they have only 5. To see this, assume that a block, say B_1, contains an independent set S of 6 vertices. Then $r_1 \in S$, and in each shaded triangle of B_1 the vertex farthest away from r_1 also belongs to S. In particular, $p_1, q_1 \in S$. Therefore, the next block, B_2, cannot have more than 4 independent vertices, one in each shaded triangle. Hence, the maximum size of an independent set of vertices in the chain C is at most $5k = 5|V(C)|/16$.

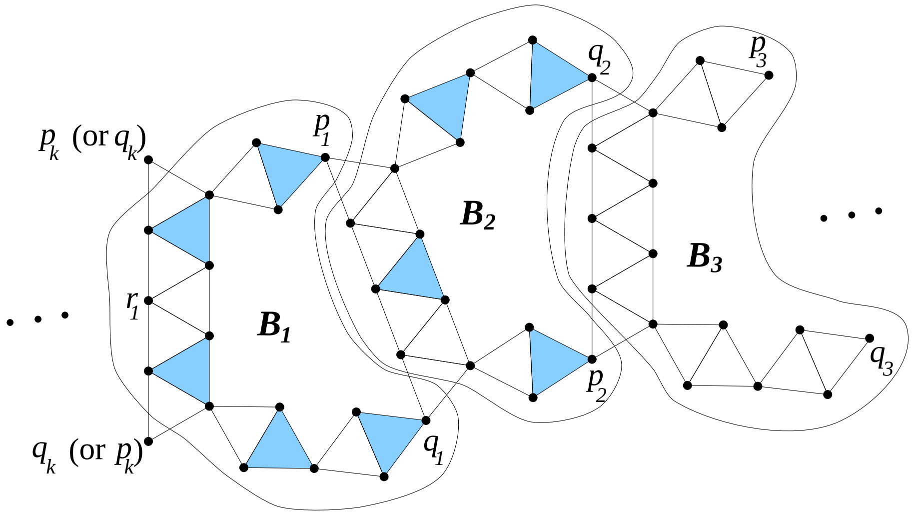

Figure 2.

If n is not a multiple of 16, then prepare a chain consisting of $\lfloor n/16 \rfloor$ blocks B_i. Arrange the remaining $n' < 16$ vertices into $\lfloor n'/3 \rfloor$ disjoint triangles and an extra edge, if necessary. $\square$

We know even less about the analogous problem in space: what is the largest number $F_3 = F_3(n)$ with the property that every system of n non-overlapping unit balls in 3-space has an independent subsystem with at least F_3 members?

Consider any system $\mathcal{B}$ of finitely many unit balls in 3-space. Pick a vertex of the convex hull of the set of centers of the balls, and let B denote the unit ball around it. Clearly, all points of incidences between B and the other members of $\mathcal{B}$ are in an open hemisphere on the surface of B. Hence, B touches at most 8 other balls [S68, S86, JR84]. It follows by induction that

$$F_3(n) \geq \left\lceil \frac{n}{9} \right\rceil .$$

On the other hand, gluing together 5 double-pyramids similarly to the planar construction shown in Figure 1, we obtain that

$$F_3(n) \leq \left\lceil \frac{5}{21} n \right\rceil .$$

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