1 FINITE POINT CONFIGURATIONS

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INTRODUCTION

The study of combinatorial properties of finite point configurations is a vast area of research in geometry, whose origins go back at least to the ancient Greeks. Since it includes virtually all problems starting with "consider a set of n points in space," space limitations impose the necessity of making choices. As a result, we will restrict our attention to Euclidean spaces and will discuss problems that we find particularly important. The chapter is partitioned into incidence problems (Section 1.1), metric problems (Section 1.2), and coloring problems (Section 1.3).

1.1 INCIDENCE PROBLEMS

In this section we will be concerned mainly with the structure of incidences between a finite point configuration P and a set of finitely many lines (or, more generally, k-dimensional flats, spheres, etc.). Sometimes this set consists of all lines connecting the elements of P. The prototype of such a question was raised by Sylvester more than one hundred years ago: Is it true that for any configuration of finitely many points in the plane, not all on a line, there is a line passing through exactly two points? The affirmative answer to this question was first given by Gallai. Generalizations for circles and conic sections in place of lines were established by Motzkin and Wilson-Wiseman, respectively.

GLOSSARY

Incidence: A point of configuration P lies on an element of a given collection of lines (k-flats, spheres, etc.).

Simple crossing: A point incident with exactly two elements of a given collection of lines.

Ordinary line: A line passing through exactly two elements of a given point configuration.

Ordinary hyperplane: A $(d-1)$-dimensional flat passing through exactly d elements of a point configuration in Euclidean d-space.

Motzkin hyperplane: A hyperplane whose intersection with a given d-dimensional point configuration lies—with the exception of exactly one point—in a $(d-2)$-dimensional flat.

Regular family of curves: A family Γ of curves in the xy-plane defined in terms of D real parameters satisfying the following properties. There is an integer s such that (a) the dependence of the curves on x, y, and the parameters

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is algebraic of degree at most s; (b) no two distinct curves of Γ intersect in more than s points; (c) for any D points of the plane, there are at most s curves in Γ passing through all of them.

Degrees of freedom: The smallest number D of real parameters defining a regular family of curves.

Spanning tree: A tree whose vertex set is a given set of points and whose edges are line segments.

Spanning path: A spanning tree that is a polygonal path.

Convex position: P forms the vertex set of a convex polygon or polytope.

k-set: A k-element subset of P that can be obtained by intersecting P with an open halfspace.

Halving plane: A hyperplane with $\lfloor |P|/2 \rfloor$ points of P on each side.

SYLVESTER-TYPE RESULTS

1. Gallai theorem (dual version): Any set of lines in the plane, not all of which pass through the same point, determines a simple crossing.

2. Motzkin-Hansen theorem: For any finite set of points in Euclidean d-space, not all of which lie on a hyperplane, there exists a Motzkin hyperplane. We obtain as a corollary that n points in d-space, not all of which lie on a hyperplane, determine at least n distinct hyperplanes. (A hyperplane is *determined* by a point set P if its intersection with P is not contained in a $(d-2)$-flat.) Putting the points on two skew lines in 3-space shows that the existence of an ordinary hyperplane cannot be guaranteed for $d > 2$.

   If $n > 8$ is sufficiently large, then any set of n noncocircular points in the plane determines at least $\binom{n-1}{2}$ distinct circles, and this bound is best possible [Ell67]. The number of ordinary circles determined by n noncocircular points is known to be at least $11n(n-1)/247$.

3. Csima-Sawyer theorem: Any set of n noncollinear points in the plane determines at least $6n/13$ ordinary lines ($n > 7$). This bound is sharp for $n = 13$ and false for $n = 7$ (see Figure 1.1.1). In 3-space, any set of n noncoplanar points determines at least $2n/5$ Motzkin hyperplanes.

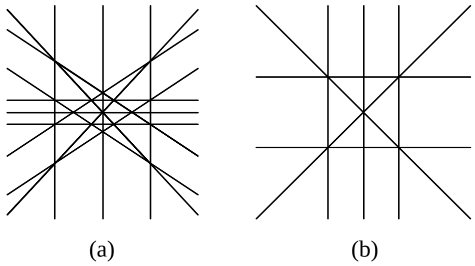


FIGURE 1.1.1
Extremal examples for the (dual) Csima-Sawyer theorem:
(a) 13 *lines (including the line at infinity) determining only* 6 *simple points;*
(b) 7 *lines determining only* 3 *simple points.*

4. Orchard problem: What is the maximum number of collinear triples determined by n points in the plane, no four on a line? There are several construc-

tions showing that this number is at least $n^2/6 - O(n)$, which is asymptotically best possible. (See Figure 1.1.2.)

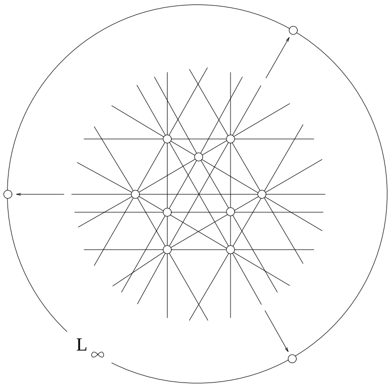


FIGURE 1.1.2
12 *points and* 19 *lines, each passing through exactly* 3 *points.*

5. Dirac's problem: Is it true that—with six exceptions listed in [Grü72]—any set of n points in the plane, not all on a line, has an element incident to at least $n/2$ connecting lines? If true, this result is best possible, as is shown by the example of n points distributed as evenly as possible on two intersecting lines. It is known that there is a positive constant c such that one can find a point incident to at least cn connecting lines. A useful equivalent formulation of this statement is that any set of n points in the plane, no more than $n-k$ of which are on the same line, determines at least $c'kn$ distinct connecting lines, for a suitable constant $c' > 0$. Note that according to the $d = 2$ special case of the Motzkin-Hansen theorem, due to Erdős (see No. 2 above), for $k = 1$ the number of distinct connecting lines is at least n. For $k = 2$, the corresponding bound is $2n - 4$, $(n \geq 10)$.

6. Ungar's theorem: n noncollinear points in the plane always determine at least $2\lfloor n/2 \rfloor$ lines of different slopes (see Figure 1.1.3); this proves Scott's conjecture. Furthermore, any set of n points in the plane, not all on a line, permits a spanning tree, all of whose $n-1$ edges have different slopes [Jam87].

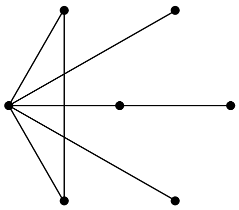


FIGURE 1.1.3
7 *points determining* 6 *distinct slopes.*

UPPER BOUNDS ON THE NUMBER OF INCIDENCES

Given a set P of n points and a family Γ of m curves or surfaces, the number of incidences between them can be obtained by summing over all $p \in P$ the number of elements of Γ passing through p. If the elements of Γ are taken from a regular family of curves with D degrees of freedom, the maximum number of incidences between P and Γ is $O(n^{D/(2D-1)}m^{(2D-2)/(2D-1)} + n + m)$. In the most important applications, Γ is a family of straight lines or unit circles in the plane ($D = 2$), or it consists of circles of arbitrary radii ($D = 3$). The best upper bounds known for the number of incidences are summarized in Table 1.1.1. It follows from the first line of the table that for any set P of n points in the plane, the number of distinct straight lines containing at least k elements of P is $O(n^2/k^3 + n/k)$, and this bound cannot be improved (Szemerédi-Trotter). In the sixth line of the table, $\beta(n, m)$ is an extremely slowly growing function, which is certainly $o(n^\epsilon m^\epsilon)$ for every $\epsilon > 0$. A collection of spheres in 3-space is said to be in *general position* here if no three of them pass through the same circle.

TABLE 1.1.1 Maximum number of incidences between n points of P and m elements of Γ. [CEG+90]

POINT SET P	FAMILY Γ	BOUND	TIGHT
Planar	lines	$O(n^{2/3}m^{2/3} + n + m)$	yes
Planar	pseudolines	$O(n^{2/3}m^{2/3} + n + m)$	yes
Planar	unit circles	$O(n^{2/3}m^{2/3} + n + m)$	?
Planar	any circles	$O(n^{3/5}m^{4/5} + n + m)$	?
Planar	pseudocircles	$O(n^{3/5}m^{4/5} + n + m)$	?
3-dimensional	spheres	$O(n^{4/7}m^{9/7}\beta(n, m) + n^2)$	?
3-dimensional	spheres in gen. position	$O(n^{3/4}m^{3/4} + n + m)$	?

MIXED PROBLEMS

Many problems about finite point configurations involve some notions that cannot be defined in terms of incidences: convex position, midpoint of a segment, etc. Below we list a few questions of this type. They are discussed in this part of the chapter, and not in Section 1.2 which deals with metric questions, because we can disregard most aspects of the Euclidean metrics in their formulation. For example, convex position can be defined by requiring that some sets should lie on one side of certain hyperplanes. This is essentially equivalent to introducing an order along each straight line.

1. Erdős-Klein-Szekeres problem: What is the maximum number of points that can be chosen in the plane so that no three are on a line and no k are in convex position ($k > 3$)? Denoting this number by $c(k)$, it is known that

$$2^{k-2} \le c(k) \le \binom{2n-4}{n-2}.$$

Let $e(k)$ denote the maximum size of a planar point set P that has no three elements on a line and no k elements that form the vertex set of an “empty” convex polygon, i.e., a convex k-gon whose interior is disjoint from P. We have $e(3) = 2$, $e(4) = 4$, $e(5) = 9$, and Horton showed that $e(k)$ is infinite for all $k \geq 7$. It is an outstanding open problem to decide whether $e(6)$ is finite.

2. The number of empty k-gons: Let $H_k^d(n)$ $(n \geq k \geq d+1)$ denote the minimum number of k-tuples that induce an empty convex polytope of k vertices in a set of n points in d-space, no $d+1$ of which lie on a hyperplane. Clearly, $H_2^1(n) = n-1$ and $H_k^1(n) = 0$ for $k > 2$. For $k = d+1$, we have

$$\frac{1}{d!} \leq \lim_{n\to\infty} H_k^d(n)/n^d \leq \frac{2}{(d-1)!},$$

[Val95]. For $d = 2$, the best estimates known for $H_k^2 = \lim_{n\to\infty} H_k^2(n)/n^2$ are

$$1 \leq H_3^2 \leq 1.68,\ 1/2 \leq H_4^2 \leq 2.42,\ 0 \leq H_5^2 \leq 1.46,$$

$$0 \leq H_6^2 \leq 1/3,\ H_7^2 = H_8^2 = \ldots = 0.$$

3. The number of k-sets: Let $N_k^d(n)$ denote the maximum number of k-sets in a set of n points in d-space, no $d+1$ of which lie on the same hyperplane. In other words, $N_k^d(n)$ is the maximum number of different ways in which k points of an n-element set can be separated from the others by a hyperplane. It is known that

$$\Omega(n \log k) \leq N_k^2(n) \leq O\left(n\sqrt{k}/\log^* k\right),$$

where $\log^* k$ denotes the iterated logarithm of k. For the number of halving planes, $N_{\lfloor n/2\rfloor}^3(n) = O(n^{8/3})$, and

$$\Omega(n^{d-1}\log n) \leq N_{\lfloor n/2\rfloor}^d(n) = o(n^d).$$

4. The number of midpoints: Let $M(n)$ denote the minimum number of different midpoints of the $\binom{n}{2}$ line segments determined by n points in convex position in the plane. One might guess that $M(n) \geq (1-o(1))\binom{n}{2}$, but it was shown in [EFF91] that

$$\binom{n}{2} - \lfloor\frac{n(n+1)(1-e^{-1/2})}{4}\rfloor \leq M(n) \leq \binom{n}{2} - \lfloor\frac{n^2-2n+12}{20}\rfloor.$$

5. Midpoint-free subsets: As a partial answer to a question proposed in [MP], it was proved by V. Bálint et al. that if $m(n)$ denotes the largest number m such that every set of n points in the plane has a midpoint-free subset of size m, then

$$\lceil\frac{-1+\sqrt{8n+1}}{2}\rceil \leq m(n) \leq o(n)$$

OPEN PROBLEMS

Here we give six problems from the multitude of interesting questions that remain open.

1. Motzkin-Dirac conjecture: Any set of n noncollinear points in the plane determines at least $n/2$ ordinary lines ($n > 13$).

2. Generalized orchard problem (Erdős): What is the maximum number of collinear k-tuples determined by n points in the plane, no $k+1$ of which are on a line ($k \geq 3$)? In particular, show that it is $o(n^2)$ for $k = 4$. The best lower bound known is $\Omega(n^{1+1/(k-2)})$.

3. Maximum independent subset problem (Erdős): Determine the largest number $\alpha(n)$ such that any set of n points in the plane, no four on a line, has an $\alpha(n)$-element subset with no collinear triples. Füredi has shown that $\Omega(\sqrt{n \log n}) \leq \alpha(n) \leq o(n)$.

4. Slope problem (Jamison): Is it true that every set of n points in the plane, not all on a line, permits a spanning path, all of whose $n-1$ edges have different slopes?

5. Empty triangle problem (Bárány): Is it true that every set of n points in the plane, no three on a line, determines at least $t(n)$ empty triangles that share a side, where $t(n)$ is a suitable function tending to infinity?

6. Balanced partition problem (Kupitz): Does there exist an integer k with the property that for every planar point set P, there is a connecting line such that the difference between the number of elements of P on its left side and right side does not exceed k? Several examples show that this assertion is not true with $k = 1$.

1.2 METRIC PROBLEMS

The systematic study of the distribution of the $\binom{n}{2}$ distances determined by n points was initiated by Erdős in 1946. Given a point configuration $P = \{p_1, p_2, \ldots, p_n\}$, let $g(P)$ denote the number of distinct distances determined by P, and let $f(P)$ denote the number of times that the unit distance occurs between two elements of P. That is, $f(P)$ is the number of pairs $p_i p_j$ $(i<j)$ such that $|p_i - p_j| = 1$. What is the minimum of $g(P)$ and what is the maximum of $f(P)$ over all n-element subsets of Euclidean d-space? These questions have raised deep number-theoretic and combinatorial problems, and have contributed richly to many recent developments in these fields.

GLOSSARY

Unit distance graph: A graph whose vertex set is a given point configuration P, in which two points are connected by an edge if and only if their distance is one.

Diameter: The maximum distance between two points of P.

General position in the plane: No three points of P are on a line, and no four on a circle.

Separated set: The distance between any two elements is at least one.

Nearest neighbor of $p \in P$: A point $q \in P$, whose distance from p is minimum.

Farthest neighbor of $p \in P$: A point $q \in P$, whose distance from p is maximum.

Homothetic sets: Similar sets in parallel position.

REPEATED DISTANCES

Extremal graph theory has played an important role in this area. For example, it is easy to see that the unit distance graph assigned to an n-element planar point set P cannot contain $K_{2,3}$, a complete bipartite graph with 2 and 3 vertices in its classes. Thus, by a well-known graph-theoretic result, $f(P)$, the number of edges in this graph, is at most $O(n^{3/2})$. This bound can be improved to $O(n^{4/3})$ by using more sophisticated combinatorial techniques (apply line 3 of Table 1.1.1 with $m = n$); but we are still far from knowing what the best upper bound is.

In Table 1.2.1, we summarize the best currently known estimates on the maximum number of times the unit distance can occur among n points in the plane, under various restrictions on their position. In the first line of the table—and throughout this chapter—c denotes (unrelated) positive constants. The second and third lines show how many times the minimum distance and the maximum distance, resp., can occur among n arbitrary points in the plane. Table 1.2.2 contains some analogous results in higher dimensions. In the first line, $\beta(n)$ is an extremely slowly growing function, closely related to the functional inverse of the Ackermann function.

TABLE 1.2.1 Estimates for the maximum number of unit distances determined by an n-element planar point set P.

POINT SET P	LOWER BOUND	UPPER BOUND	SOURCE
Arbitrary	$n^{1+c/\log\log n}$	$O(n^{4/3})$	Erdős, Spencer et al.
Separated	$\lfloor 3n - \sqrt{12n-3} \rfloor$	$\lfloor 3n - \sqrt{12n-3} \rfloor$	Reutter, Harborth
Of diameter 1	n	n	Hopf-Pannwitz
In convex position	$2n-7$	$O(n \log n)$	Edelsbrunner-Hajnal, Füredi
No 3 collinear	$\Omega(n \log n)$	$O(n^{4/3})$	Kárteszi
Separated, no 3 coll.	$(2 + 5/16 - o(1))n$	$(2+3/7)n$	[Tót95]

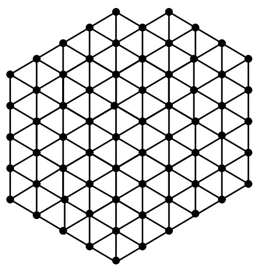


FIGURE 1.2.1
A separated point set with $\lfloor 3n - (12n-3)^{1/2} \rfloor$ unit distances ($n = 69$). All such sets have been characterized by Kupitz.

TABLE 1.2.2 Estimates for the maximum number of unit distances determined by an n-element point set P in d-space.

POINT SET P	LOWER BOUND	UPPER BOUND	SOURCE
$d=3$, arbitrary	$\Omega(n^{4/3}\log\log n)$	$O(n^{3/2}\beta(n))$	Clarkson et al.
$d=3$, separated	$6n-O(n^{2/3})$	$6n-\Omega(n^{2/3})$	Newton
$d=3$, diameter 1	$2n-2$	$2n-2$	Grünbaum, Heppes
$d=3$, on sphere (rad. $1/\sqrt{2}$)	$\Omega(n^{4/3})$	$O(n^{4/3})$	Erdős-Hickerson-Pach
$d=3$, on sphere (rad. $r\neq 1/\sqrt{2}$)	$\Omega(n\log^* n)$	$O(n^{4/3})$	Erdős-Hickerson-Pach
$d>3$ even, arb.	$\frac{n^2}{2}\left(1-\frac{1}{\lfloor d/2\rfloor}\right)+n-O(d)$	$\frac{n^2}{2}\left(1-\frac{1}{\lfloor d/2\rfloor}\right)+n-\Omega(d)$	Erdős
$d>3$ odd, arb.	$\frac{n^2}{2}\left(1-\frac{1}{\lfloor d/2\rfloor}\right)+\Omega(n^{4/3})$	$\frac{n^2}{2}\left(1-\frac{1}{\lfloor d/2\rfloor}\right)+O(n^{4/3})$	Erdős-Pach

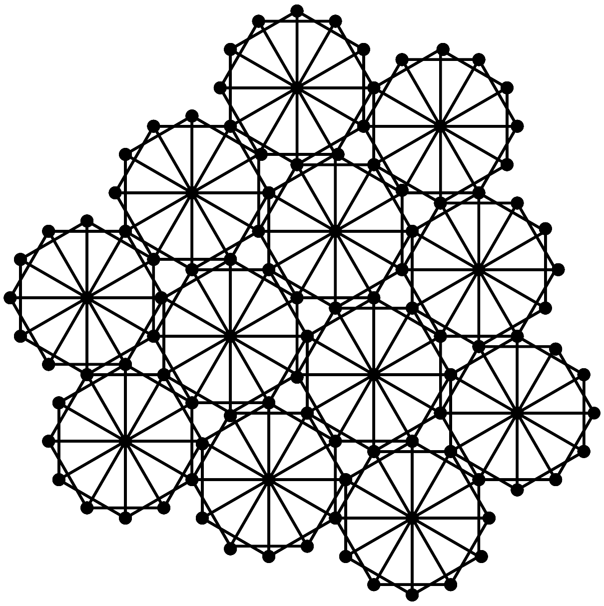


FIGURE 1.2.2
n points, among which the second-smallest distance occurs $(\frac{24}{7}+o(1))n$ times.

The second line of Table 1.2.2 can be extended by showing that the smallest distance cannot occur more than $3n-2k+4$ times between points of an n-element set in the plane whose convex hull has k vertices. The maximum number of occurrences of the second-smallest and second-largest distance is $(24/7+o(1))n$ and $3n/2$ (if n is even), respectively (Brass, Vesztergombi).

Given any point configuration P, let $\Phi(P)$ denote the sum of the numbers of farthest neighbors for every element $p\in P$. Table 1.2.3 contains tight upper bounds on $\Phi(P)$ in the plane and in 3-space, and asymptotically tight ones for higher dimensions [ES89], [Csi95], [EP90].

TABLE 1.2.3 Upper bounds on $\Phi(P)$, the total number of farthest neighbors of all points of an n-element set P.

POINT SET P	UPPER BOUND
Planar, n is even	$3n-3$
Planar, n is odd	$3n-4$
Planar, in convex position	$2n$
3-dimensional, $n \equiv 0 \pmod 2$	$n^2/4+3n/2+3$
3-dimensional, $n \equiv 1 \pmod 4$	$n^2/4+3n/2+9/4$
3-dimensional, $n \equiv 3 \pmod 4$	$n^2/4+3n/2+13/4$
d-dimensional ($d>3$)	$n^2(1-1/\lfloor d/2 \rfloor + o(1))$

DISTINCT DISTANCES

It is obvious that if all distances between pairs of points of a d-dimensional set P are the same, then $|P| \le d+1$. If P determines at most g distinct distances, we have that $|P| \le \binom{d+g}{d}$; see [BBS83]. This implies that if d is fixed and n tends to infinity, then the minimum number of distinct distances determined by n points in d-space is at least $\Omega(n^{1/d})$. Denoting this minimum by $g_d(n)$, for $d \ge 3$ we have the following results:

$$\Omega(n^{1/(d-1)}/2^{c\alpha^2(n)}) \le g_d(n) \le O(n^{2/d}),$$

where $\alpha(n)$ is the (extremely slowly growing) functional inverse of Ackermann's function. In Table 1.2.4, we list some lower and upper bounds on the minimum number of distinct distances determined by an n-element point set P, under various assumptions on its structure.

TABLE 1.2.4 Estimates for the minimum number of distinct distances determined by an n-element point set P in the plane.

POINT SET P	LOWER BOUND	UPPER BOUND	SOURCE
Arbitrary	$\Omega(n^{4/5})$	$O(n/\sqrt{\log n})$	Székely [Szé95]
In convex position	$\lfloor n/2 \rfloor$	$\lfloor n/2 \rfloor$	Altman
No 3 collinear	$\lceil (n-1)/3 \rceil$	$\lfloor n/2 \rfloor$	Szemerédi
In general position	$\Omega(n)$	$O(n^{1+c/\sqrt{\log n}})$	Erdős, Füredi et al.

RELATED RESULTS

1. Integer distances: There are arbitrarily large, noncollinear finite point sets in the plane such that all distances determined by them are integers, but there exists no infinite set with this property.

2. Generic subsets: Any set of n points in the plane contains $\Omega(n^{1/4})$ points such that all distances between them are distinct. This bound could perhaps be improved to about $n^{1/3}$; see [LT95].

3. Borsuk's problem: It was conjectured that every (finite) d-dimensional point set P can be partitioned into $d+1$ parts of smaller diameter. It follows from the results quoted in the third lines of Tables 1.2.1 and 1.2.2 that this is true for $d = 2$ and 3. Surprisingly, Kahn and Kalai proved that there exist sets P that cannot be partitioned into fewer than $(1.2)^{\sqrt{d}}$ parts of smaller diameter. In particular, the conjecture is false for $d = 946$. On the other hand, it is known that for large d, every d-dimensional set can be partitioned into $(\sqrt{3/2} + o(1))^d$ parts of smaller diameter [Sch88].

4. Nearly equal distances: Two numbers are said to be nearly equal if their difference is at most one. If n is sufficiently large, then the maximum number of times that nearly the same distance occurs among n separated points in the plane is $\lfloor n^2/4 \rfloor$. The maximum number of pairs in a separated set of n points in the plane, whose distance is nearly equal to any one of k arbitrarily chosen numbers, is $\frac{n^2}{2}(1 - \frac{1}{k+1} + o(1))$, as n tends to infinity [EMP93].

5. Repeated angles: In an n-element planar point set, the maximum number of noncollinear triples that determine the same angle is $O(n^2 \log n)$, and this bound is asymptotically tight (Pach-Sharir). The corresponding maximum in 3-space is at most $O(n^{8/3})$, but in 4-space the angle $\pi/2$ can occur $\Omega(n^3)$ times (Croft, Purdy).

6. Repeated triangles: Let $t_d(n)$ denote the maximum number of triples in an n-element point set in d-space that induce a unit area triangle. It is known that $\Omega(n^2 \log\log n) \le t_2(n) \le O(n^{7/3})$, $t_5(n) = o(n^3)$, and $t_6(n) = \Theta(n^3)$ (Pach-Sharir, Purdy). In the plane, the maximum number of triples that determine a triangle of unit perimeter, or an isosceles triangle, is also $O(n^{7/3})$.

7. Similar triangles: There exists a positive constant c such that for any triangle T and any $n \ge 3$, there is an n-element point set in the plane with at least cn^2 triples that induce triangles similar to T. For most quadrilaterals Q, the maximum number of 4-tuples of an n-element set that induce quadrilaterals similar to Q is $o(n^2)$. The maximum number of pairwise homothetic triples in a set of n points in the plane is $O(n^{3/2})$, and this bound is asymptotically tight [EE94].

CONJECTURES OF ERDŐS

1. The number of times the unit distance can occur among n points in the plane does not exceed $n^{1+c/\log\log n}$.

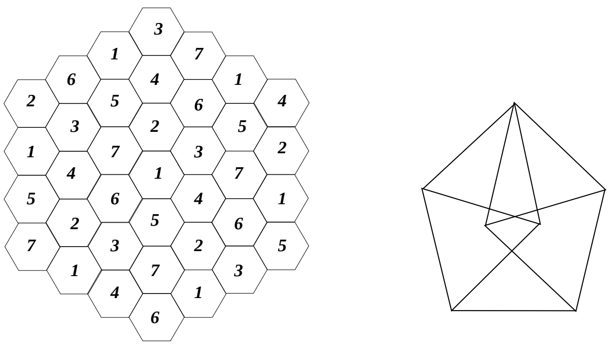
2. Any set of n points in the plane determines at least $\Omega(n/\sqrt{\log n})$ distinct distances.

3. Any set of n points in convex position in the plane has a point from which there are at least $\lfloor n/2 \rfloor$ distinct distances.

4. There is an integer $k \geq 4$ such that any finite set in convex position in the plane has a point from which there are no k points at the same distance.
5. Any set of n points in the plane, not all on a line, contains at least $n - 2$ triples that determine distinct angles (Corrádi, Erdős, Hajnal).
6. The diameter of any set of n points in the plane with the property that the set of all distances determined by them is separated (on the line) is at least $\Omega(n)$. Perhaps it is at least $n - 1$, with equality when the points are collinear.

1.3 COLORING PROBLEMS

If we partition a space into a small number of parts (i.e., we color its points with a small number of colors), at least one of these parts must contain certain "unavoidable" point configurations. In the simplest case, the configuration consists of a pair of points at a given distance. The prototype of such a question is the Hadwiger-Nelson problem: What is the minimum number of colors needed for coloring the plane so that no two points at unit distance receive the same color? The answer is known to be between 4 and 7.



(i) (ii)

FIGURE 1.3.1
The chromatic number of the plane is (i) *at most* 7 *and* (ii) *at least* 4.

GLOSSARY

Chromatic number of a graph: The minimum number of colors, $\chi(G)$, needed to color all the vertices of G so that no two vertices of the same color are adjacent.

List-chromatic number of a graph: The minimum number k such that for any assignment of a list of k colors to every vertex of the graph, for each vertex it is possible to choose a single color from its list so that no two vertices adjacent to each other receive the same color.

Chromatic number of a metric space: The chromatic number of the unit distance graph of the space, i.e., the minimum number of colors needed to color all points of the space so that no two points of the same color are at unit distance.

Polychromatic number of metric space: The minimum number of colors, χ, needed to color all points of the space so that for each color class C_i, there is a distance d_i such that no two points of C_i are at distance d_i. A sequence of "forbidden" distances, $(d_1, \ldots, d_\chi)$, is called a ***type*** of the coloring. (The same coloring may have several types.)

Girth of a graph: The length of the shortest cycle in the graph.

A point configuration P is ***k-Ramsey*** in d-space if, for any coloring of the points of d-space with k colors, at least one of the color classes contains a congruent copy of P.

A point configuration P is ***Ramsey*** if, for every k, there exists $d(k)$ such that P is k-Ramsey in $d(k)$-space.

Brick: The vertex set of a right parallelepiped.

FORBIDDEN DISTANCES

Table 1.3.1 contains the best bounds we know for the chromatic numbers of various spaces. All lower bounds can be established by showing that the corresponding unit distance graphs have some *finite* subgraphs of large chromatic number. $S^{d-1}(r)$ denotes the sphere of radius r in d-space, where the distance between two points is the length of the chord connecting them.

TABLE 1.3.1 Estimates for the chromatic numbers of metric spaces.

SPACE	LOWER BOUND	UPPER BOUND	SOURCE
Line	2	2	
Plane	4	7	Nelson, Isbell
Rational points of plane	2	2	Woodall
3-space	5	21	Raĭskiĭ
Rational points of 3-space	2	2	Benda, Perles
$S^2(r), \frac{1}{2} \le r \le \frac{\sqrt{3-\sqrt{3}}}{2}$	3	4	Simmons
$S^2(r), \frac{\sqrt{3-\sqrt{3}}}{2} \le r \le \frac{1}{\sqrt{3}}$	3	5	Straus
$S^2(r), r \ge \frac{1}{\sqrt{3}}$	4	7	Simmons
$S^2\left(\frac{1}{\sqrt{2}}\right)$	4	4	Simmons
Rational points of 4-space	4	4	Benda, Perles
Rational points of 5-space	6	?	Chilakamarri
d-space	$(1+o(1))(1.2)^d$	$(3+o(1))^d$	Frankl-Wilson, Larman-Rogers
$S^{d-1}(r), r \ge \frac{1}{2}$	d	?	Lovász

Next we list several problems and results strongly related to the Hadwiger-Nelson problem (quoted in the introduction to this section).

1. Polychromatic number: Stechkin and Woodall showed that the polychromatic number of the plane is between 4 and 6. It is known that for any $r \in [\sqrt{2}-1, 1/\sqrt{5}]$, there is a coloring of type $(1,1,1,1,1,r)$ [Soi94]. However, the list-chromatic number of the unit distance graph of the plane, which is at least as large as its polychromatic number, is infinite.

2. Dense sets realizing no unit distance: The *lower* (resp. *upper*) *density* of an unbounded set in the plane is the lim inf (resp. lim sup) of the ratio of the Lebesgue measure of its intersection with a disk of radius r around the origin to $r^2\pi$, as $r \to \infty$. If these two numbers coincide, their common value is called the ***density*** of the set. Let δ^d denote the maximum density of a planar set, no pair of points of which is at unit distance. Croft showed that $0.2293 \le \delta^2 \le 0.2857$.

3. The graph of large distances: Let $G_i(P)$ denote the graph whose vertex set is a finite point set P, with two vertices connected by an edge if and only if their distance is one of the i largest distances determined by P. In the plane, $\chi(G_1(P)) \le 3$ for every P; see Borsuk's problem in the preceding section. It is also known that for any finite planar set, $G_i(P)$ has a vertex with fewer than $3i$ neighbors (Erdős-Lovász-Vesztergombi). Thus, $G_i(P)$ has fewer than $3in$ edges, and its chromatic number is at most $3i$. However, if $n > ci^2$ for a suitable constant $c > 0$, we have $\chi(G_i(P)) \le 7$.

EUCLIDEAN RAMSEY THEORY

According to an old result of Gallai, for any finite d-dimensional point configuration P and for any coloring of d-space with finitely many colors, at least one of the color classes will contain a homothetic copy of P. The corresponding statement is false if, instead of a homothet, we want to find a *translate*, or even a *congruent copy*, of P. Nevertheless, for some special configurations, one can establish interesting positive results, provided that we color a sufficiently high-dimensional space with a sufficiently small number of colors. The Hadwiger-Nelson-type results discussed in the preceding subsection can also be regarded as very special cases of this problem, in which P consists of only two points. The field, known as "Euclidean Ramsey theory", was started by a series of papers by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus.

For details, see Chapter 8 of this Handbook.

OPEN PROBLEMS

1. (Erdős, Simmons) Is it true that the chromatic number of $S^{d-1}(r)$, the sphere of radius r in d-space, is equal to $d+1$, for every $r > 1/2$? In particular, does this hold for $d = 3$ and $r = 1/\sqrt{3}$?

2. (Erdős) Does there exist an integer g such that the chromatic number of any unit distance graph in the plane whose girth is at least g does not exceed 3? It is known that if such an integer exists, it must be at least 5 [Wor79].

3. (Sachs) What is the minimum number of colors, $\chi(d)$, sufficient to color any system of nonoverlapping unit balls in d-space so that no two balls that are tangent to each other receive the same color? Equivalently, what is the maximum chromatic number of a unit distance graph induced by a d-dimensional separated point set? It is easy to see that $\chi(2) = 4$, and we also know that $5 \leq \chi(3) \leq 9$.

4. (Ringel) Does there exist any finite upper bound on the number of colors needed to color any system of (possibly overlapping) disks (of not necessarily equal radii) in the plane so that no two disks that are tangent to each other receive the same color, provided that no three disks touch one another at the same point? If such a number exists, it must be at least 5.

5. (Graham) Is it true that any 3-element point set P that does not induce an equilateral triangle is 2-Ramsey in the plane? This is known to be false for equilateral triangles, and correct for right triangles (Shader). Is every 3-element point set P 3-Ramsey in 3-space? The answer is again in the affirmative for right triangles (Bóna and Tóth).

1.4 SOURCES AND RELATED MATERIAL

SURVEYS

All results not given an explicit reference above may be traced in these surveys.

[PA95]: A monograph devoted to combinatorial geometry.

[Pac93]: A collection of essays covering a large area of discrete and computational geometry, mostly of some combinatorial flavor.

[HDK64]: A classical treatise of problems and exercises in combinatorial geometry, complete with solutions.

[KW91]: A collection of beautiful open questions in geometry and number theory, together with some partial answers organized into challenging exercises.

[EP95]: A survey full of original problems raised by the "founding father" of combinatorial geometry.

[JT95]: A collection of more than two hundred unsolved problems about graph colorings, with an extensive list of refererences to related results.

[Grü72]: A monograph containing many results and conjectures on configurations and arrangements.

RELATED CHAPTERS

Chapter 4: Helly-type theorems and geometric transversals
Chapter 5: Pseudoline arrangements

Chapter 8: Euclidean Ramsey theory
Chapter 10: Geometric discrepancy theory and uniform distribution
Chapter 11: Topological methods
Chapter 21: Arrangements

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