Geometric graph theory

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Summary A *geometric graph* is a graph drawn in the plane such that its vertices are points in general position and its edges are straight-line segments. The study of geometric graphs is a fairly new discipline abounding in open problems, but it has already yielded some striking results that have proved to be instrumental for the solution of various problems in combinatorial and computational geometry. These include the k-set problem, proximity questions, bounding the number of incidences between points and lines, designing various efficient graph drawing algorithms, etc. This paper surveys some Turán-type and Ramsey-type extremal problems for geometric graphs, and discusses their generalizations and applications.

1 Introduction, basic definitions

Let G be a finite graph with no loops or multiple edges, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. By a *drawing* of G, we mean a representation of G in the plane such that each vertex is represented by a distinct point and each edge by a simple (non-self-intersecting) continuous arc connecting the corresponding two points. If it is clear that we are referring to a drawing, and not to the underlying "abstract" graph, these points and arcs will also be called *vertices* and *edges*, respectively.

Two edges (arcs) *cross* each other if they have an interior point in common. This point is called a *crossing*. A crossing p is called *proper* if in a small neighbourhood of p one edge passes from one side of the other edge to the other side. We assume throughout that in a drawing

1. no edge passes through any vertex other than its endpoints,
2. no two edges touch each other (i.e., if two edges cross, then they properly cross),
3. no three edges cross at the same point.

A drawing in which every edge is represented by a straight-line segment is called a *straight-line drawing* or a *geometric graph*. Two geometric graphs are *isomorphic* if and only if one of them can be carried into the other using a rigid motion of the plane. We always assume that the vertices of a geometric graph are in *general position*, i.e. no three of them are collinear. If the vertices of a geometric graph are in *convex position*, i.e. they form the vertex set of a convex polygon, then the graph is called a *convex geometric graph*.

We emphasize that two edges of a geometric graph G may cross each other. The *number of crossings* in G is denoted by $\mathrm{CR}(G)$. Two edges of G are *disjoint* if they do not cross and do not even share an endpoint.

A geometric graph G is said to be *complete* if its edge set consists of all $\binom{|V(G)|}{2}$ segments between its vertices. If $V(G) = V_1 \cup V_2$, and $E(G)$ consists of all segments between V_1 and V_2, then G is a *complete bipartite* geometric graph. A geometric graph G is said to be a *geometric tree* if the underlying abstract graph is a tree. A geometric graph G is a *geometric subgraph* of another geometric graph H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

The systematic study of geometric graphs was initiated by S. Avital and H. Hanani [7], P. Erdős, Y. Kupitz [48], and M. Perles. They realized that many classical questions in extremal graph theory [10] have natural analogues for geometric graphs. Some of these questions turned out to be surprisingly difficult and required new techniques combining geometric and combinatorial tools. In this paper we survey some recent results of this type and some tantalizing open problems.

Section 2 focuses on crossing-free geometric graphs. Section 3 describes some conditions which guarantee that a geometric graph determines many crossings. Sections 4,5 and 6 concentrate on Turán-type and Ramsey-type problems for geometric graphs, respectively. In Section 7 we present a few applications of our results, while Section 8 is devoted to geometric hypergraphs.

As usual, for any positive integer-valued functions, $f(n)$ and $g(n)$, we use the notations $f(n) = o(g(n)), f(n) = O(g(n))$, and $f(n) = \Omega(g(n))$ to indicate that $\lim_{n\to\infty} f(n)/g(n) = 0$, $\sup f(n)/g(n) < \infty$, and $\inf f(n)/g(n) > 0$, respectively. We write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

For many other problems and results in this area, see [56] and Chapter 14 of [57].

2 Crossing-free geometric graphs

Let $G = (V(G), E(G))$ be a geometric graph which has no crossing. Clearly, the underlying abstract graph, G_0, whose vertex set is $V(G)$ and whose edge set consists of those pairs of vertices which are connected in G by a segment, is *planar*. Of course, the reverse is not true: the fact that G_0 is planar does not imply that G must be crossing-free. However, by Fáry's theorem, for every abstract planar graph H, there is a crossing-free geometric graph G whose underlying graph G_0 is isomorphic to H. In simpler terms, we have the following

Theorem 2.1 *[24], [78], [71]. Every planar graph admits a crossing-free straight-line drawing.*

We can modify the problem by putting certain restrictions on the drawing. For instance, given an abstract planar graph H with n vertices and a set P of n points in general position in the plane, we can ask whether there exists a crossing-free geometric graph G with $V(G) = P$, whose underlying graph G_0 is isomorphic to H. To see that the answer is not necessarily in the affirmative, choose H to be a complete graph with 4 vertices, and let P consist of 4 points in convex position.

An abstract graph H is called *outerplanar* if it can be drawn in the plane without crossing so that all points representing the vertices of H lie on the boundary of the outer face of the resulting subdivision of the plane. Clearly, every tree is outerplanar. It follows from Theorem 2.1 that a graph is outerplanar if and only if it is the underlying graph of a crossing-free *convex* geometric graph (a triangulated cycle). However, these graphs also satisfy a stronger condition.

Theorem 2.2 *[30]. For any outerplanar graph H with n vertices and for any set P of n points in the plane in general position, there is a crossing-free geometric graph G with $V(G) = P$, whose underlying graph is isomorphic to H.*

In other words, every outerplanar graph H admits a straight-line drawing without crossing such that its vertex set is mapped into an arbitrarily prespecified $|V(H)|$-element set. Often we do not have much freedom in how to select this mapping. But sometimes we do. Perles conjectured that if H is a tree, then we have at least one degree of freedom. More precisely, the following assertion is true.

Theorem 2.3 *[63], [37]. Let T be a rooted tree, and let P be a set of $|V(T)|$ points in the plane in general position with a specified element $p \in P$.*

Then T admits a crossing-free straight-line drawing such that every vertex of T is represented by an element of P, and the root is represented by p.

It may be conjectured that for any forest F consisting of k rooted trees $T_1, T_2, \ldots, T_k$ with roots $r_1, r_2, \ldots, r_k$, and for any set $P = \{p_1, p_2, \ldots, p_K\}$ of $K = \sum_{i=1}^{k} |T_k|$ points in general position in the plane, F admits a crossing-free straight-line drawing such that every vertex of F is represented by a point of P, and r_i is represented by p_i $(i = 1, 2, \ldots, k)$. A. Kaneko and M. Kano [39], [40], [41] verified this conjecture in several special cases, including when

1. $k = 2$;
2. every tree T_i is a *rooted star*, i.e., all of its vertices are adjacent to the root r_i;
3. all trees T_i are of the same size.

However, Kaneko and Kano believe that the above conjecture, in general, is false.

In some special cases, we may impose further restrictions on the drawings. For instance, colour the vertices of the tree T with red and blue such that no two vertices of the same colour are adjacent. Let n_1 and n_2 denote the number of red and blue vertices, respectively. Given an n_1-element set P_1 and an n_2-element set P_2 in general position in the plane, we wish to decide whether there is a crossing-free straight-line drawing of T, whose red and blue vertices are represented by the elements of P_1 and P_2, respectively. In the case when P_1 and P_2 are separated by a straight-line, $n_1 = n_2$, and T is a path, the answer is in the affirmative [35].

Abellanas, García, Hernández, Noy and Ramos studied the following relaxation of this question: what happens if, instead of embedding a fixed tree T, we want to find a drawing of at least one member of a given *class* of trees?

Problem 2.4 *[1]. Let $P_1 \cup P_2$ be a point set in general position in the plane, $|P_1| \geq |P_2| > 0$. Is it true that there always exists a crossing-free geometric tree on the vertex set $P_1 \cup P_2$, whose every edge runs between P_1 and P_2, and in which the degree of every vertex is at most an absolute constant times $|P_1|/|P_2|$?*

Abellanas et al. showed that the answer is in the affirmative if

1. $P_1 \cup P_2$ is in convex position, or

2. P_1 and P_2 are separated by a straight line.

We close this section with a related question.

Problem 2.5 *[25]. What is the size of the smallest point set P in the plane with the property that for every abstract planar graph H of n vertices, there is a crossing-free geometric graph G, whose vertex set is a subset of P and whose underlying graph is isomorphic to H?*

The next theorem shows that this minimum is $O(n^2)$ (see also [68]).

Theorem 2.6 *[25]. For every planar graph H, there is a crossing-free geometric graph whose vertex set is a subset of an n by $2n$ piece of the integer grid, and whose underlying graph is isomorphic to H.*

3 Unavoidable crossings

The number of crossings in a geometric graph G is denoted by $\text{CR}(G)$. It follows from Euler's Polyhedral Formula that a geometric graph with n vertices and more than $3n-6$ edges cannot be crossing-free ($n \geq 3$). In fact, this easily implies the following quantitative statement.

Lemma 3.1 *Any geometric graph G with $n \geq 3$ vertices and e edges has at least $e - 3n + 6$ crossings.*

Proof By induction on e. If $e \le 3n-6$ then the statement is void. Assume $e > 3n-6$. Then G has at least one crossing. Applying the induction hypothesis to the graph G' obtained from G by deleting an edge which participates in at least one crossing, we obtain that

$$\text{CR}(G) = 1 + \text{CR}(G') \le 1 + [(e-1) - 3n + 6)] = e - 3n + 6. \quad \blacksquare$$

It is easy to see that Lemma 3.1 cannot be improved if $3n-6 \le e \le 4n-8$. However, as was shown by R. Eggleton and R. Guy [20], if $e > 4n-8$ then we can also find an edge which participates in at least *two* crossings. Using this fact, the above inductional argument yields that G has at least

$$[e - (3n-6)] + [e - (4n-8)] = 2e - 7n + 14$$

crossings.

This argument can be pushed a little further. Pach and G. Tóth proved the following.

Theorem 3.2 *[65]. Let G be a geometric graph with $n \ge 3$ vertices and e edges such that every edge crosses at most k others. Then, for $0 \le k \le 4$, we have*

$$e \le (k+3)(n-2).$$

For $0 \le k \le 2$, this bound cannot be improved.

As before, this leads to a further improvement of Lemma 3.1.

Corollary 3.3 *[65]. Any geometric graph G with $n \ge 3$ vertices and e edges has at least*

$$\sum_{k=0}^{4} [e - (k+3)(n-2)] = 5e - 25n + 50$$

crossings. That is, we have $\text{CR}(G) \ge 5e - 25 + 50$.

Ajtai, Chvátal, Newborn and Szemerédi [3] and independently T. Leighton [51] discovered that from Lemma 3.1 (or from the slightly stronger Corollary 3.3), by a simple trick, one can deduce a much better lower bound for the number of crossings in 'dense' graphs.

Theorem 3.4 *[3], [51], [65]. Let G be a geometric graph with n vertices and e edges. Then $\text{CR}(G)$, the number of crossings in G, satisfies*

$$\text{CR}(G) \ge \frac{1}{33.75}\frac{e^3}{n^2} - 0.9n > 0.029\frac{e^3}{n^2} - 0.9n.$$

Proof Suppose first that $e \geq 7.5n$. Construct a *random* geometric subgraph $G' \subseteq G$ by selecting each vertex of G independently with probability $p = 7.5n/e \leq 1$, and letting G' be the geometric subgraph induced by the selected vertices. The expected number of vertices of G', $E[n'] = pn$. Similarly, the expected number of edges and the expected number of crossings in G' satisfy $E[e'] = p^2 e$ and $E[\text{CR}\,(G')] = p^4\,\text{CR}\,(G)$, respectively.

Applying Corollary 3.3 to G' and taking expectations, we obtain

$$p^4\,\text{CR}\,(G) = E[\text{CR}\,(G')] > 5E[e'] - 25E[n'] = 5p^2 e - 25pn.$$

This implies that

$$\text{CR}\,(G) \geq \frac{1}{33.75}\frac{e^3}{n^2},$$

whenever $e \geq 7.5n$. In fact, using Corollary 3.3 in the range $e < 7.5n$, it is easy to check that the slightly weaker inequality in Theorem 3.4 is valid for every geometric graph G. ■

Theorem 3.4 does not remain true if we replace the constant $1/33.75 \approx 0.029$ by 0.06. To see this, let $n \ll e \ll n^2$, and consider a geometric graph whose vertex set can be obtained by a slight perturbation of a $\lfloor\sqrt{n}\rfloor$ by $\lfloor\sqrt{n}\rfloor$ piece of the integer grid, and connect two gridpoints, p and q, by an edge if and only if their distance satisfies

$$|p - q| \leq \sqrt{\frac{2}{\pi}\cdot\frac{e}{n}}.$$

It is not hard to show that the number of crossings in this geometric graph is at most $0.06e^3/n^2$.

One may believe that extending Theorem 3.2 to larger values of k would result in a further improvement of the constant factor in Theorem 3.4. However, this is not the so. Although the estimate in Theorem 3.2 is tight for small values of k, it becomes weak as k increases. It follows from Theorem 3.4 that if every edge of a geometric graph with n vertices and e edges participates in at most $k \geq 1$ crossings, then

$$e \leq \sqrt{16.875kn} \approx 4.108\sqrt{k}n, \tag{1}$$

and this bound is tight apart from the value of the constant.

Although Theorem 3.4 is tight up to a constant factor, somewhat stronger estimates can be obtained for some special classes of geometric graphs. Let $\mathcal{K}_{r,s}$ denote the class of geometric graphs consisting of all rs segments between some r-element set and some s-element set in the plane. In other words, $\mathcal{K}_{r,s}$ is the family of all *complete bipartite* geometric graphs with r and s vertices in their vertex classes.

Theorem 3.5 *[61]. Let G be a geometric graph with n vertices and $e \geq 4n$ edges, which does not contain a complete bipartite geometric subgraph belonging to $\mathcal{K}_{r,s}$.*

Then the number of crossings in G satisfies

$$\text{CR}(G) \geq c_{r,s} \frac{e^{3+1/(r-1)}}{n^{2+1/(r-1)}},$$

where $c_{r,s} > 0$ is a suitable constant. These bounds are tight up to a constant factor, whenever $s > (r-1)!$.

We close this section with another useful lower bound on the number of crossings in a geometric graphs. To state this result, we need a definition.

Definition Let G_0 be an *abstract* graph with vertex set $V(G_0)$ and edge set $E(G_0)$. The *bisection width*, $b(G_0)$, of G_0 is defined as the minimum number of edges, whose removal splits the graph into two roughly equal subgraphs. More precisely, $b(G_0)$ is the minimum number of edges running between V_1 and V_2, over all partitions of the vertex set of G_0 into two disjoint parts $V_1 \cup V_2$ such that $|V_1|, |V_2| \geq |V(G_0)|/3$.

The *bisection width* of a *geometric* graph G is defined as the bisection width of the underlying abstract graph G_0, and is denoted by $b(G)$.

Leighton observed that there is an intimate relationship between the bisection width and the number of crossings in a geometric graph [51], which is based on the Lipton–Tarjan separator theorem for planar graphs [52]. The following version of this relationship was obtained by Pach, F. Shahrokhi, and M. Szegedy.

Theorem 3.6 *[58]. Let G be a geometric graph of n vertices with degrees $d_1, d_2, \ldots, d_n$. Then*

$$\text{CR}\,(G) \geq \frac{1}{40} b^2(G) - \frac{1}{16} \sum_{i=1}^{n} d_i^2.$$

4 Forbidden geometric subgraphs – Multiple crossings

In the spirit of Turán's classical theorem for abstract graphs [77], one can raise the following general question. Given a class $\mathcal{H}$ of so-called *forbidden geometric subgraphs*, what is the maximum number of edges that a geometric graph (resp. a convex geometric graph) of n vertices can have without containing a geometric subgraph which belongs to $\mathcal{H}$? Denote this maximum by $\text{ex}\,(\mathcal{H}, n)$ (resp. by $\text{ex}_c(\mathcal{H}, n)$).

For any $k \geq 2$, let $\mathcal{C}_k$ denote the class of all geometric graphs consisting of k pairwise *crossing* edges.

The simple fact that a crossing-free geometric graph of n vertices has at most $3n-6$ edges, can now be expressed by

$$\text{ex}\,(\mathcal{C}_2, n) = 3n - 6,$$

for every $n \geq 3$. Similarly, inequality (1) can be rewritten as

$$\text{ex}\,(\mathcal{X}_{1,k}, n) \leq 4.108\sqrt{k}n,$$

where $\mathcal{X}_{1,k}$ denotes the class of all geometric graphs consisting of $k+1$ edges such that one of them crosses the other k. (Note that $\mathcal{X}_{1,1} = \mathcal{C}_2$.)

The order of magnitude of the function $\text{ex}\,(\mathcal{C}_3, n)$ has been determined by Agarwal, Aronov, Pach, Pollack and Sharir.

Theorem 4.1 *[2]. The maximum number of edges that a geometric graph of n vertices can have without containing 3 pairwise crossing edges satisfies*

$$\text{ex}\,(\mathcal{C}_3, n) = O(n).$$

For $k \geq 4$, we do not know whether $\text{ex}\,(\mathcal{C}_k, n)$ is linear in n. However, as was shown by Pach, Shahrokhi and Szegedy, one can apply Theorems 3.6 and 4.1 to deduce that $\text{ex}\,(\mathcal{C}_k, n)$ is bounded from above by n times a polylogarithmic factor.

Corollary 4.2 *[58]. For every $k \geq 3$, we have*

$$\text{ex}\,(\mathcal{C}_k, n) = O(n \log^{2k-6} n).$$

Proof The assertion is true for $k = 3$. We sketch the proof only for $k = 4$. The other cases can be treated similarly.

Let G be a geometric graph of n vertices with degrees $d_1, d_2, \ldots, d_n$. Suppose that G has no 4 pairwise crossing edges and $|E(G)| = \text{ex}\,(\mathcal{C}_4, n)$. For every $e \in E(G)$, the edges crossed by e form a geometric graph with no 3 pairwise crossing edges. Therefore, by Theorem 4.1, every edge crosses at most $O(n)$ other edges, and $\text{CR}\,(G) = O(|E(G)|n)$. Since

$$\sum_{i=1}^{n} d_i^2 \leq \left(\sum_{i=1}^{n} d_i\right)\left(\max_{1\leq i\leq n} d_i\right) \leq 2|E(G)|n,$$

it follows from Theorem 3.6 that $b(G) = O\left(\sqrt{|E(G)|n}\right)$.

Consider a partition of $V(G)$ into two parts, V_1 and V_2, such that $|V_1| = n_1 \geq n/3$, $|V_2| = n_2 \geq n/3$, and the number of edges connecting them is $b(G)$. Let G_1 and G_2 denote the geometric subgraphs of G induced by V_1 and V_2, respectively. Since neither G_1 nor G_2 contains 4 pairwise crossing edges, we obtain

$$|E(G)| = |E(G_1)| + |E(G_2)| + b(G) \leq \text{ex}\,(\mathcal{C}_4, n_1) + \text{ex}\,(\mathcal{C}_4, n_2) + O(|E(G)|n),$$

which yields by an easy induction argument that

$$|E(G)| = \text{ex}\,(\mathcal{C}_4, n) = O(n \log^2 n). \quad \blacksquare$$

Using the theory of generalized Davenport–Schinzel sequences [45], P. Valtr proved the following.

Theorem 4.3 *[79]. Let G be a geometric graph with n vertices, all of whose edges can be intersected by a line. If G has no k pairwise crossing edges, then its number of edges is at most $c_k n$, where c_k is a suitable constant depending only on k.*

This result immediately implies that for every $k \geq 4$, the exponent $2k-6$ of the logarithmic factor in Corollary 4.2 can be reduced to 1.

Corollary 4.4 *[79]. For every $k \geq 4$, we have*

$$\mathrm{ex}\,(\mathcal{C}_k, n) = O(n \log n).$$

Proof Let G be a geometric graph with n vertices, with no k pairwise crossing edges, and suppose that $|E(G)| = \mathrm{ex}\,(\mathcal{C}_k, n)$. Take a vertical line l which does not pass through any of the vertices and which partitions the vertex set of G into two roughly equal parts, V_1 and V_2, with $|V_1| = \lfloor n/2 \rfloor$ and $|V_2| = \lceil n/2 \rceil$. Let G_1 and G_2 denote the geometric subgraphs of G induced by G_1 and G_2, respectively, and let G' be the geometric graph formed by the edges of G intersecting l. Clearly, the number of edges of G is equal to

$$\mathrm{ex}\,(\mathcal{C}_k, n) = |E(G_1)| + |E(G_2)| + |E(G')| \leq \mathrm{ex}\,(\mathcal{C}_k, \lfloor n/2 \rfloor) + \mathrm{ex}\,(\mathcal{C}_k, \lceil n/2 \rceil) + c_k n,$$

where we used Theorem 4.3 to estimate $|E(G')|$. The result follows. ■

It is conjectured that, for every fixed k, $\mathrm{ex}\,(\mathcal{C}_k, n)$ is linear in n. One can also make the following stronger conjecture.

Conjecture 4.5 *For any $k \geq 3$, there exists an integer C_k with the property that the edges of every geometric graph which has no k pairwise crossing edges can be coloured by C_k colours so that no two edges of the same colour cross each other.*

A somewhat weaker conjecture which would also imply the linearity of $\mathrm{ex}\,(\mathcal{C}_k, n)$ is the following. For any $k \geq 3$, there exists a positive constant γ_k such that every geometric graph without k pairwise crossing edges has a crossing-free geometric subgraph which has at least $\gamma_k |E(G)|$ edges.

Unfortunately, we do not even know if Conjecture 4.5 is true for $k = 3$. S. McGuinness [54] settled the case $k = 3$ for every geometric graph whose edges can be intersected by a straight line. Conjecture 4.5 is also known to be true for convex geometric graphs. The following theorem of A. Kostochka and J. Kratochvíl slightly improves an old bound of A. Gyárfás [31],[32] (see also [33]) and Kostochka [46].

Theorem 4.6 *[47]. Let G be a convex geometric graph which has no k pairwise crossing edges. Then one can colour the edges of G with 2^{k+5} colours in such a way that no two edges of the same colour cross each other.*

Kostochka [46] constructed a series of convex geometric graphs with no k pairwise crossing edges, which cannot be coloured with fewer than a constant times $k \log k$ colours.

Theorem 4.6 plays an important role in the proof of Theorem 4.1. It also implies that $\mathrm{ex}_c(\mathcal{C}_k, n) \leq 2^{k+5}(2n-3) = O(n)$ for every fixed k. However, V. Capoyleas and Pach determined the exact value of $\mathrm{ex}_c(\mathcal{C}_k, n)$, which is one of the few non-trivial precise results in the area.

Theorem 4.7 *[13]. Let $n \geq 2k$. The maximum number of edges that a convex geometric graph of n vertices can have without containing k pairwise crossing edges satisfies*

$$\mathrm{ex}_c(\mathcal{C}_k, n) = 2(k-1)n - \binom{2k-1}{2}.$$

All of the results presented in this section can be rephrased in the following way. If a geometric graph G with n vertices has sufficiently many edges, then it must have a k-tuple of pairwise crossing edges. In fact, if $|E(G)|$ is very large, there are many such k-tuples. In the previous section we described this phenomenon for $k = 2$ by a fairly tight quantitative formula. The proof of Theorem 3.4 may serve as a prototype of how to obtain many similar results.

Theorem 4.8 *[55]. For any $k \geq 2$ there exists a constant $c(k) > 0$ such that every convex geometric graph with n vertices and $e \geq (2k-1)n$ edges has at least $c(k)e^{2k-1}/n^{2k-2}$ k-tuples of pairwise crossing edges.*

For $k = 2$ and 3, the last statement can be extended to all (not necessarily convex) geometric graphs. The same is true for every fixed k such that $\mathrm{ex}(\mathcal{C}_k, n) = O(n)$. By Theorem 4.8, if G has constant times n^2 edges, then the total number of k-tuples of edges in G is at least $c'n^{2k}$, for some $c' > 0$. A positive per cent of them are pairwise crossing.

5 Forbidden geometric subgraphs – Non-crossing configurations

In the previous section we discussed how many edges guarantee the existence of k pairwise crossing edges in a geometric graph. Here we address the 'dual' question where, in place of pairwise crossing edges we look for pairwise *disjoint* edges. Recall that two disjoint edges cannot share even an endpoint.

For any $k \geq 2$, let $\mathcal{D}_k$ denote the class of all geometric graphs consisting of k pairwise *disjoint* edges.

The following observation of Kupitz settles the question for convex geometric graphs.

Theorem 5.1 *[49]. Let* $\mathrm{ex}_c(\mathcal{D}_k, n)$ *denote the maximum number of edges that a convex geometric graph of* n *vertices can have without containing* k *pairwise disjoint edges. Then for every* k *and* $n \geq 2k$*, we have*

$$\mathrm{ex}_c(\mathcal{D}_k, n) = (k-1)n.$$

Proof Let G be a convex geometric graph with n vertices. Assume without loss of generality that $V(G)$ is the vertex set of a regular n-gon. Partition $E(G)$ into n classes so that two segments belong to the same class if and only if they are parallel. If G has no k pairwise crossing edges then each class contains at most $k-1$ elements. Hence, $|E(G)| \leq (k-1)n$, and this bound can be attained. ■

The case of non-convex geometric graphs is more difficult. Hopf and Pannwitz [**?**] and (independently) Erdős proved a long time ago the following

Theorem 5.2 *[36]. Let* $\mathrm{ex}\,(\mathcal{D}_2, n)$ *denote the maximum number of edges that a geometric graph with* n *vertices can have without containing two disjoint edges. Then, for every* $n \geq 3$,

$$\mathrm{ex}\,(\mathcal{D}_2, n) = n.$$

Proof (M. Perles) We say that an edge xy is to the *left of* xz if the ray xz can be obtained from the ray xy by a clockwise turn of less than π. A vertex x is called *pointed* if all edges incident to it lie in a half-plane whose boundary contains x.

For every pointed vertex of G, delete the leftmost edge incident with it. If G had no two disjoint edges then the resulting graph is empty. Indeed, if there were at least one remaining edge, xz, then we could find an edge xy to the left of xz and another edge zv to the left of $zx = xz$, and these two edges would be disjoint. Since we deleted at most one edge for each vertex, $|E(G)| \leq |V(G)|$, and this bound can be attained. ■

Alon and Erdős [5] were the first to show the linearity of $\mathrm{ex}\,(\mathcal{D}_3, n)$. The best known lower and upper bounds,

$$2.5n - 4 \leq \mathrm{ex}\,(\mathcal{D}_3, n) \leq 3n$$

are due to Perles (unpublished) and Goddard, Katchalski and Kleitman [28], respectively. Recently, Tóth and Valtr [76] showed that

$$4n - 9 \leq \mathrm{ex}\,(\mathcal{D}_4, n) \leq 8.5n.$$

Pach and Törőcsik proved that $\mathrm{ex}\,(\mathcal{D}_k, n) = O(n)$, for every fixed k.

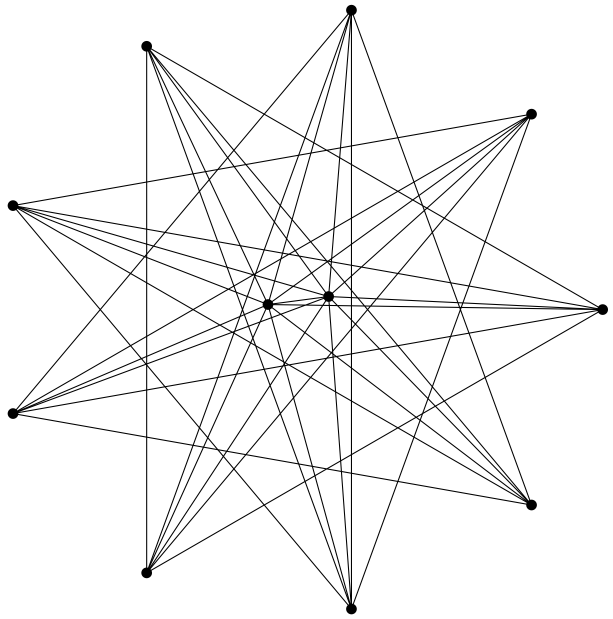

Figure 1: A geometric graph with no 4 pairwise disjoint edges.

Theorem 5.3 *[64]. Let* $\mathrm{ex}\,(\mathcal{D}_k, n)$ *denote the maximum number of edges that a geometric graph of* n *vertices can have without containing* k *pairwise disjoint edges. Then, for every* k *and* n*, we have*

$$\mathrm{ex}\,(\mathcal{D}_k, n) \leq (k-1)^4 n.$$

Proof Let G be a geometric graph with n vertices, containing no k pairwise disjoint edges. For any vertex v, let $x(v)$ and $y(v)$ denote the x-coordinate and the y-coordinate of v, respectively. Assume without loss of generality that no two vertices have the same x-coordinate.

An edge $e \in E(G)$ is said to lie *below* another edge $e' \in E(G)$ if there is no vertical line which intersects both e and e', and whose intersection with e' is strictly below its intersection with e. (According to this definition, if the projections of e and e' onto the x-axis do not overlap, then we say that e lies below e' and also e' lies below e.)

Define four binary relations $\prec_1$, $\prec_2$, $\prec_3$, and $\prec_4$ on $E(G)$, as follows. One necessary condition for two edges to be in any of these relations is that they must be disjoint. Let $e = v_1v_2$ and $e' = v_1'v_2'$ be two disjoint edges of G with $x(v_1) < x(v_2)$ and $x(v_1') < x(v_2')$. Then

- $e \prec_1 e'$ if $x(v_1) < x(v_1'),\ x(v_2) < x(v_2')$ and e lies below e';
- $e \prec_2 e'$ if $x(v_1) > x(v_1'),\ x(v_2) > x(v_2')$ and e lies below e';
- $e \prec_3 e'$ if $x(v_1) < x(v_1'),\ x(v_2) > x(v_2')$ and e lies below e';

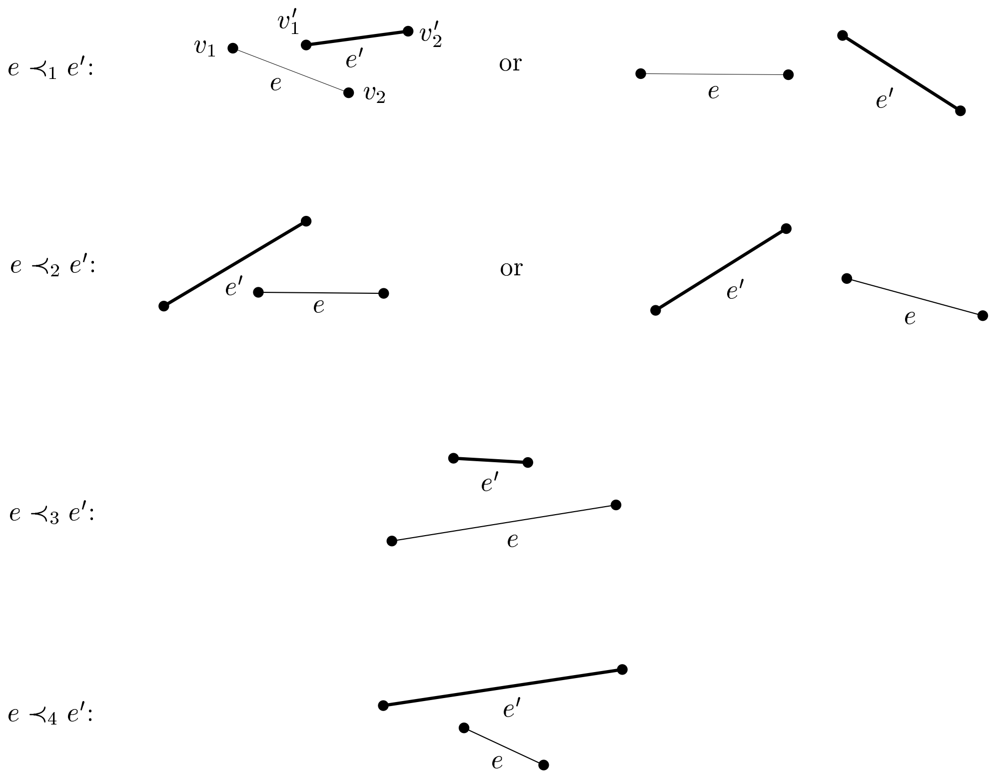

Figure 2: The relations $\prec_i$.

- $e \prec_4 e'$ if $x(v_1) > x(v_1')$, $x(v_2) < x(v_2')$ and e lies below e'.

Obviously, each of the relations $\prec_i$ defines a partial order on $E(G)$, and any pair of disjoint edges is comparable by at least one of them. Since G has no k pairwise disjoint edges, there is no chain of length k in the partially ordered set $(E(G), \prec_i)$ $(1 \leq i \leq 4)$. This implies that, for each i, $E(G)$ can be partitioned into at most $k-1$ classes (*antichains*) such that no two edges belonging to the same class are comparable by $\prec_i$. Overlaying these four partitions, we obtain a decomposition of $E(G)$ into at most $(k-1)^4$ classes E_j $(1 \leq j \leq (k-1)^4)$ such that no two elements of E_j are comparable by any of the relations $\prec_i$. Thus, none of the graphs $G_j := (V(G), E_j)$ has two disjoint edges. In view of Theorem 5.2, $|E(G_j)| = |E_j| \leq n$ $(1 \leq j \leq (k-1)^4)$, and the result follows. ■

Recently, Tóth improved the above result by showing that the dependence of $\mathrm{ex}\,(\mathcal{D}_k, n)$ on k is at most quadratic. Indeed, he proved a somewhat stronger result.

The *length* of a path is the number of its edges. Let $\mathcal{P}_k$ denote the class of all non-crossing paths of length k. Clearly, every non-crossing path of length $2k-1$ has k pairwise disjoint edges.

Theorem 5.4 *[75]. Let* $\mathrm{ex}\,(\mathcal{P}_k, n)$ *denote the maximum number of edges that a geometric graph of* n *vertices can have without containing a non-crossing path of length* k*. Then for every* k *and* n*, we have*

$$\mathrm{ex}\,(\mathcal{P}_k, n) \leq Ck^2 n,$$

for a suitable constant C*.*

It is possible, however, that the true order of magnitude of $\mathrm{ex}\,(\mathcal{P}_k, n)$ is $O(kn)$. According to a result of Perles, this holds for convex geometric graphs.

Theorem 5.5 *[44]. Let* $\mathrm{ex}_c(\mathcal{P}_k, n)$ *denote the maximum number of edges that a convex geometric graph of* n *vertices can have without containing a non-crossing path of length* k*. Then for every* $n \geq k+1$*, we have*

$$\mathrm{ex}_c(\mathcal{P}_k, n) = \lfloor \frac{(k-1)n}{2} \rfloor.$$

A graph that can be obtained by attaching some edges to the vertices of a path is called a *caterpillar*. In other words, a caterpillar is a tree which contains no three paths of length 2 meeting at a common endpoint. We say that two geometric graphs are of the same *type* if their underlying abstract graphs are isomorphic. In its most general form, Perles' (unpublished) result is the following.

Theorem 5.6 *Let* $\mathcal{CP}_k$ *denote the class of all non-crossing caterpillars of a fixed type, consisting of* k *edges. Then for every* $n \geq k+1$*, we have*

$$\mathrm{ex}_c(\mathcal{CP}_k, n) = \lfloor \frac{(k-1)n}{2} \rfloor.$$

We mention a related result for (not necessarily convex) geometric graphs, which can easily be deduced from Theorem 2.2 and from Turán's theorem quoted at the beginning of this section.

Corollary 5.7 *Let* $\mathcal{OP}_k$ *denote the class of all non-crossing outerplanar graphs of a fixed type, which can be obtained from a cycle of length* k *by adding some (or no) diagonals. Then for every* $n \geq k+1$*, we have*

$$\mathrm{ex}\,(\mathcal{OP}_k, n) = \mathrm{ex}_c(\mathcal{OP}_k, n) = \frac{k-2}{2(k-1)}(n^2 - r^2) + \binom{r}{2},$$

where r *is the remainder of* n *upon division by* $k-1$*.*

As we pointed out at the end of the preceding section, whenever $\mathrm{ex}\,(\mathcal{H}, n) = O(n)$ holds for a class of forbidden geometric subgraphs $\mathcal{H}$, we can use the idea of the proof of Theorem 3.4 to show that, if $|E(G)| = e \gg \mathrm{ex}\,(\mathcal{H}, n)$, then G has many subgraphs belonging to $\mathcal{H}$. In particular, Theorem 5.3 can be used to establish the following.

Theorem 5.8 *[55]. For any $k \geq 2$ there exists a constant $c(k) > 0$ such that every geometric graph with n vertices and $e \geq k^4 n$ edges has at least $c(k)e^{2k-1}/n^{2k-2}$ k-tuples of pairwise disjoint edges.*

Theorem 5.4 implies that the last statement also holds with the weaker assumption $e \geq Ck^2 n$.

6 Ramsey-type results

In classical Ramsey–theory, one wants to find large monochromatic subgraphs in a complete graph whose edges are coloured with several colours [12],[29]. Most questions of this type can be generalized to complete geometric graphs, where the monochromatic subgraphs are required to satisfy certain geometric conditions.

The next two statements were conjectured by Bialostocki and Dierker and proved by Károlyi, Pach and Tóth.

Theorem 6.1 *[42],[43]. If the edges of a finite complete geometric graph are coloured by two colours, there exists a non-crossing spanning tree, all of whose edges are of the same colour.*

Proof Let $V = \{v_1, v_2, \ldots, v_n\}$ denote the vertex set of a complete geometric graph K_n whose edges are coloured with red and blue. Suppose without loss of generality that no two vertices have the same x-coordinate and that the vertices are listed in increasing order of their x-coordinates. The assertion is trivial for $n \leq 2$. Thus, we can assume that $n \geq 3$ and the theorem has already been proved for all complete geometric graphs having fewer than n vertices. We can also assume that all edges along the boundary of the convex hull of V are of the same colour (say, red). Indeed, if two consecutive edges of the convex hull have different colours, then remove their common endpoint from K_n. By the induction hypothesis, the remaining graph has a monochromatic non-crossing spanning tree. Depending on its colour, this spanning tree can be completed to a monochromatic non-crossing spanning tree of K_n, by putting back one of the two previously deleted edges of the convex hull of V.

For every i, $1 < i < n$, let K_i^l and K_i^r denote the subgraphs of K_n induced by the points $\{v_1, \ldots, v_i\}$ and $\{v_i, \ldots, v_n\}$, respectively. By the induction hypothesis, both K_i^l and K_i^r have a monochromatic non-crossing spanning tree, T_i^l resp. T_i^r. We can assume that these two trees have different colours, otherwise their union will meet the requirements of the theorem. We can also assume that T_2^l is red and T_2^r is blue. Otherwise, T_2^r would be red, and it could be completed to a non-crossing red spanning tree of K_n by the addition of any edge of the convex hull of V incident to v_1. Similarly, we can suppose that T_{n-1}^l is blue and T_{n-1}^r is red. Hence, there exists an i, $1 < i < n-1$, such that

- T_i^l is red and T_i^r is blue,

- T^l_{i+1} is blue and T^r_{i+1} is red.

Connecting T^l_i and T^r_{i+1} by any edge of the convex hull of V which intersects a vertical line separating v_i and v_{i+1}, we obtain a non-crossing red spanning tree of K_n, as required. ■

Theorem 6.2 *[42],[43]. If the edges of a complete geometric graph with $3n-1$ vertices are coloured by two colours, there exist n pairwise disjoint edges of the same colour. This result is best possible.*

The analogues of Theorems 6.1 and 6.2 for *abstract* graphs, i. e., when the geometric constraints are ignored, were found by Erdős–Rado (see [22]) and Gerencsér and Gyárfás [27], respectively. In fact, Gerencsér and Gyárfás proved the stronger result that for any 2-colouring of the edges of a complete graph with $3n-1$ vertices, there exists a monochromatic path of length $2n-1$. This statement, as well as Theorem 6.2, is best possible, as is shown by the following example. Take the disjoint union of a complete graph of $n-1$ vertices and a complete graph of $2n-1$ vertices, all of whose edges are red and blue, respectively, and colour all edges between the two parts red.

Theorem 6.2 also has an "asymmetric" version.

Theorem 6.3 *[42],[43]. Let k and l be positive integers, $n = k+l+\max\{k,l\}-1$. If the edges of a complete geometric graph with n vertices are coloured by red and blue, one can find either k disjoint red edges or l disjoint blue edges. This result cannot be improved.*

To formulate some further results of this type, it is convenient to introduce some notation.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be not necessarily different classes of geometric graphs. Let $R(\mathcal{G}_1, \mathcal{G}_2)$ denote the smallest positive number R with the property that any complete geometric graph of R vertices, whose edges are coloured with red and blue, contains a geometric subgraph which is either totally red and belongs to $\mathcal{G}_1$ or totally blue and belongs to $\mathcal{G}_2$. If $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}$, we write $R(\mathcal{G})$ instead of $R(\mathcal{G}_1, \mathcal{G}_2)$. If we restrict our attention to *convex* geometric graphs, then the corresponding function is denoted by $R_c(\mathcal{G})$. Clearly, $R_c(\mathcal{G}) \le R(\mathcal{G})$.

There is an obvious relationship between $R(\mathcal{G})$ and the function $\mathrm{ex}(\mathcal{G}, n)$ discussed in the preceding sections:

$$\mathrm{ex}(\mathcal{G}, n) < \frac{1}{2}\binom{n}{2} \implies R(\mathcal{G}) \le n.$$

Indeed, if we colour the edges of a complete geometric graph with two colours, one of the colour classes will contain at least $\frac{1}{2}\binom{n}{2}$ elements. If $\mathrm{ex}(\mathcal{G}, n) < \frac{1}{2}\binom{n}{2}$, then there is a geometric graph belonging to $\mathcal{G}$, all of whose edges are in this colour class. Similarly, we have

$$\mathrm{ex}_c(\mathcal{G}, n) < \frac{1}{2}\binom{n}{2} \implies R_c(\mathcal{G}) \le n. \tag{2}$$

In the special case when $\mathcal{G}$ is the class $\mathcal{T}_n$ of all non-crossing geometric trees of n vertices and the class $\mathcal{D}_n$ of all geometric graphs consisting of n disjoint edges, respectively, the values of $R(\mathcal{G})$ are given in Theorems 6.1 and 6.2. Theorem 6.3 provides the exact value of $R(\mathcal{G}_1, \mathcal{G}_2)$, when $\mathcal{G}_1 = \mathcal{D}_k$ and $\mathcal{G}_2 = \mathcal{D}_l$.

$$R(\mathcal{T}_n) = R_c(\mathcal{T}_n) = n, \quad R(\mathcal{D}_n) = R_c(\mathcal{D}_n) = 3n - 1,$$

$$R(\mathcal{D}_k, \mathcal{D}_l) = R_c(\mathcal{D}_k, \mathcal{D}_l) = k + l + \max\{k, l\} - 1.$$

Next we show

Theorem 6.4 *[44]. Let $\mathcal{CC}_n$ and $\mathcal{CC}'_n$ denote, respectively, the class of all non-crossing cycles of length n and the class of all non-crossing cycles of length n triangulated from a vertex. Then we have*

$$(n-1)^2 < R_c(\mathcal{CC}_n) \le R(\mathcal{CC}'_n) \le 2(n-1)(n-2) + 2.$$

Proof First we prove that $R_c(\mathcal{CC}_n) > (n-1)^2$. Take $(n-1)^2$ points on a circle and partition them into $n-1$ groups, each containing $n-1$ consecutive points. Color red all edges between points in different groups, and colour blue all edges between points belonging to the same group.

Any non-crossing red cycle contains at most one point from each group; hence it cannot have more than $n-1$ points. On the other hand, all vertices of a blue cycle are from the same group, so there is no blue cycle with more than $n-1$ points.

Next we show that $R(\mathcal{CC}'_n) \le 2(n-1)(n-2)+2$. Let V denote the vertex set of a complete geometric graph K of $2(n-1)(n-2)+2$ vertices, whose edges are coloured with red and blue. Let v be a vertex of the convex hull of V. Since there are $2(n-1)(n-2)+1$ edges incident to v, at least $(n-1)(n-2)+1$ of them are of the same colour, say, red. Let $v_1, v_2, \ldots, v_{(n-1)(n-2)+1}$ be the vertices of K, listed in clockwise order of visibility from v, such that each edge vv_i is red. A path $v_{i_1}v_{i_2}\ldots v_{i_j}$ is said to be *monotone* if $i_1 < i_2 < \ldots < i_j$.

Define a partial ordering of the vertices $v_1, v_2, \ldots, v_{(n-1)(n-2)+1}$, as follows. Let $v_i < v_j$ if $i < j$ and there is a monotone red path connecting v_i to v_j. As in the proof of Theorem 5.3, there are either $n-1$ elements that form a chain (totally ordered subset), or n elements that form an antichain (i.e. they are pairwise incomparable). In the former case, there is a monotone red path $u_1, u_2, \ldots, u_{n-1}$, and we can complete it to a non-crossing red cycle $v, u_1, u_2, \ldots, u_{n-1}$ of size n, together with the corresponding diagonals from v. In the latter case, there is a complete blue subgraph of n vertices, $u_1, u_2, \ldots, u_n$, because any two incomparable elements are connected by a blue edge. By Theorem 2.2, this complete blue subgraph contains a non-crossing cycle of length n, triangulated from a vertex. ■

Corollary 6.5 *[44]. For any 2-colouring of the edges of a complete geometric graph with n vertices, there exist monochromatic non-crossing cycles of length $3, 4, \ldots, \lfloor\sqrt{n/2}\rfloor$, having the same colour. The order of magnitude of this bound cannot be improved.*

Let $\mathcal{P}_n$ denote the class of all non-crossing paths of length n. It is an immediate consequence of the above results that

$$R(\mathcal{P}_n) \leq R(\mathcal{CC}_{n+1}) \leq 2n(n-1)+2.$$

However, we have a better upper bound on $R(\mathcal{P}_n)$.

Theorem 6.6 *[44]. For any 2-colouring of the edges of a complete geometric graph with n vertices, there exists monochromatic non-crossing path whose length is at least a constant times $n^{2/3}$. Equivalently,*

$$R(\mathcal{P}_n) = O(n^{3/2}).$$

It is very likely that $R(\mathcal{P}_n)$ is linear in n. For convex geometric graphs we can verify this conjecture.

Theorem 6.7 *[44]. For any 2-colouring of the edges of a convex complete geometric graph with $n \geq 3$ vertices, there exists monochromatic non-crossing path of length $\lfloor\frac{n+1}{2}\rfloor$. This bound cannot be improved. Consequently, for every n, we have*

$$R_c(\mathcal{P}_n) = 2n-1. \tag{3}$$

Proof We prove only the easy part of (3): if n is even then $R_c(\mathcal{P}_n) \leq 2n-1$. In view of (2) and Theorem 5.6, it is enough to verify that

$$\lfloor\frac{(n-1)(2n-1)}{2}\rfloor < \frac{1}{2}\binom{2n-1}{2},$$

which is trivially true. ■

Some other Ramsey numbers for convex geometric graphs were studied by H. Harborth and H. Lefmann [34].

For any positive integer n, let $n\mathcal{G}$ denote the class of all geometric graphs that can be obtained by taking the union of n pairwise disjoint members of $\mathcal{G}$, any two of which can be separated by a straight line.

Theorem 6.8 *[44]. Let $\mathcal{G}$ be any class of geometric graphs, each of which has at least two vertices.*

1. *If n is a power of 2 then*

$$R(n\mathcal{G}) \leq (R(\mathcal{G})+1)n-1.$$

2. For any $n > 0$,

$$R(n\mathcal{G}) \le \left\lceil \frac{3(R(\mathcal{G})+1)}{2} \right\rceil n - \left\lceil \frac{R(\mathcal{G})+1}{2} \right\rceil .$$

3. For any $n > 0$,

$$R_c(n\mathcal{G}) \le (R_c(\mathcal{G})+1)n - 1 .$$

In particular, if $\mathcal{G} = \mathcal{D}_1$ is the class of all geometric graphs consisting of a single edge, then $R(\mathcal{D}_1) = 2$, and part 1 of Theorem 6.8 implies that

$$R(n\mathcal{D}_1) = R(\mathcal{D}_n) \le 3n - 1,$$

provided that n is a power of 2. Theorem 6.2 shows that this inequality holds for every n, and it is best possible.

Suppose next that $\mathcal{G} = \Delta (= \mathcal{CC}_3)$ is the class of all triangles. Then we have $R(\Delta) = 6$, so, by part 1 of Theorem 6.8, now we obtain

$$R(n\Delta) \le 7n - 1,$$

provided that n is a power of 2. The following theorem shows that this result cannot be improved.

Theorem 6.9 *[44] Let Δ denote the class of triangles and let n be a positive integer. Then*

$$R(n\Delta) \ge (R(\Delta)+1)n - 1 = 7n - 1 .$$

7 Applications

This section outlines a few fairly direct applications of the results presented before. L. Székely [72] discovered that the the following celebrated result of Szemerédi and Trotter readily follows from Theorem 3.4.

Corollary 7.1 *[73], [74]. The maximum number of incidences between m distinct lines and n distinct points in the plane is at most*

$$O\left(m^{2/3}n^{2/3} + m + n\right).$$

This bound is tight up to a constant factor.

Proof Assume without loss of generality that every line is incident to at least one point. Let I denote the number of incidences between our points and lines. Define a geometric graph G whose vertex set is the given set of n points, and whose two vertices are connected by a segment if they are consecutive vertices along one of the lines. Clearly, $|E(G)| = I - m$.

On the other hand, by Theorem 3.4, either $|E(G)| < 4n$ or

$$\text{CR}(G) \ge \frac{1}{33.75}\frac{(I-m)^3}{n^2}.$$

The result follows by comparing this bound with the trivial relation $\text{CR}(G) \le \binom{m}{2}$. ■

The same idea was explored by T. Dey [16] to achieve a breakthrough in the so-called k-set problem mentioned in the abstract, which plays an important role in the analysis of many algorithms in computational geometry.

Given a set P of n points in the plane in general position (no 3 on a line), a k-element subset $P' \subseteq P$ is said to be a *k-set*, if there is an open half-plane H such that $H \cap P = P'$. It was shown by Erdős, Lovász, Simmons and Straus [23], [53] that the number of k-sets of an n-element set is $O(n^{3/2})$, and this was slightly improved by Pach, Steiger and Szemerédi [62]. The order of magnitude of the best known lower bound is $n \log n$.

Theorem 7.2 *[16]. For every $0 < k < n$, the number of k-sets of an n-element point set in general position in the plane is at most $6.5(k+1)^{1/3}n$.*

Proof Let P be a fixed set of n points in the plane in general position. Let H_k be a geometric graph defined on the vertex set P such that two points $u, v \in P$ are connected by a segment if and only if there are precisely $k-1$ points of P strictly on one side of the line uv. It is easy to see that the number of edges of H_k is equal to the number of k-sets of P. Orient every edge of H_k from left to right, and assume without loss of generality that at least $|E(H_k)|/2$ edges have precisely $k-1$ points on their *left-hand* side. Let G_k denote the graph formed by these edges.

Let the *successor* of an edge $\vec{uv} \in E(G_k)$ be the edge $\vec{vw}$ with minimum slope among all edges $\vec{vw'}$ whose slopes exceed that of $\vec{uv}$. This relation arranges the edges of G_k into pairwise disjoint convex chains. It is not hard to verify that there is precisely one chain starting (ending) at each of the k leftmost (rightmost) points of P, and no chain starts (ends) anywhere else.

Assign to each crossing $p \notin V(G)$ of two chains, γ_1 and γ_2, the uniquely determined segment $u_p v_p$ below p, whose supporting line is a common tangent to γ_1 and γ_2. (We say that a segment is *below* p if it intersects the half-line starting at p and pointing to the direction of the negative y-axis.) Clearly, $u_p, v_p \in P$, and no segment is assigned to two different crossings. The number of tangent segments assigned to crossings is at most kn, because each vertex occurs at most once as the left endpoint of a tangent segment, whose other endpoint lies on a chain not passing through p. Thus, the total number of crossings,

$$\mathrm{CR}(G_k) \leq nk.$$

On the other hand, by Theorem 3.4, either $|E(G_k)| < 4n$ or

$$\mathrm{CR}(G_k) \geq \frac{1}{33.75} \frac{|E(G_k)|^3}{n^2}.$$

Comparing these two bounds, we obtain

$$|E(H_k)| \leq 2|E(G_k)| \leq 6.5(k+1)^{1/3}n. \quad \blacksquare$$

The proofs of Theorems 3.4 and 3.6 easily generalize to arbitrary graph drawings, where the arcs representing the edges are not necessarily straight-line segments. Using these generalizations, the next two theorems, due to Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [15] and Spencer, Szemerédi and Trotter [70], can be established by minor modifications of the proof of Corollary 7.1.

Corollary 7.3 *[15], [18]. The total number of sides of m distinct cells in an arrangement of n lines in the plane is at most $O(m^{2/3}n^{2/3}+m)$.*

Corollary 7.4 *[70].The number of unit distances determined by n points in the plane is $O(n^{4/3})$.*

A family Γ of curves in the plane is said to have d *degrees of freedom* if there exists an integer s such that

- no two curves in Γ have more than s points in common, and
- for any d points, there are at most s curves in Γ passing through all of them.

Pach and Sharir applied the above mentioned (straightforward) extension of Theorem 3.4 to obtain the following common generalization of Corollaries 7.1 and 7.4.

Theorem 7.5 *[60], [59]. Let Γ be a family of curves in the plane with d degrees of freedom. Then the maximum number of incidences between n points in the plane and m elements of Γ is*

$$O(n^{d/(2d-1)}m^{(2d-2)/(2d-1)}+n+m).$$

Using the same technique, Székely slightly improved a well-known result of Chung, Szemerédi and Trotter [14].

Theorem 7.6 *[72]. Any set of n points in the plane has an element, from which there are at least $cn^{4/5}$ distinct distances to the remaining points. (Here $c>0$ is an absolute constant.)*

Next we present some easy consequences of Theorem 3.6 and its extension to arbitrary graph drawings. The proof of Corollary 4.2 was also based on this result.

Corollary 7.7 *The number of crossings in an arbitrary drawing of a random graph with n vertices and $e\geq 4n$ edges is at least Ce^2, where $C>0$ is a suitable constant. This bound is evidently tight, apart from the constant.*

Proof It is sufficient to verify that the bisection width of such a random graph is almost surely at least constant times e. ■

A drawing of a graph is called *polygonal* if every edge is represented by a non-crossing polygonal path. In Section 2, we saw that every outerplanar graph of n vertices admits a straight-line drawing without crossing such that its vertex set is mapped into an arbitrarily prespecified n-element set (Theorem 2.2). If we also want to specify the location of each vertex, then usually we cannot insist on straight-line edges. Of course, even in this case we can always find a crossing-free *polygonal* drawing. We want to minimize the total length (i.e. the total number of segments) of the polygonal paths representing the edges.

Corollary 7.8 *[66]. For every n, there exist an abstract outerplanar graph G_n of n vertices and an assignment of locations (distinct points) for the vertices such that in any planar drawing of G_n with the property that each vertex is mapped into the point assigned to it and each edge is represented by a polygonal path, there are at least $n/100$ edges consisting of at least $n/100$ segments.*

Proof We show only the weaker assertion that there exist a G_n and an assignment of locations such that the total number of segments forming the edges of G_n in any polygonal drawing of G_n with the given point locations, is at least constant times n^2.

Let G_n be a matching of size $\lfloor n/2 \rfloor$, and let the set of locations be the vertex set P of a regular n-gon. Randomly assign to each vertex of G_n a distinct element of P. Assume that with this assignment the matching can be realized with pairwise disjoint polygonal paths of total length L. Add to this drawing the edges of the convex hull of P. In this way, we produce a polygonal drawing of a graph G'_n obtained from a cycle by adding a random set of $\lfloor n/2 \rfloor$ vertex-disjoint diagonals. As every edge of a polygonal path crosses at most two convex hull edges, the number of crossings in this drawing is at most $2L$. On the other hand, it is easy to see that, almost surely, the bisection width of G'_n is at least Cn, for a suitable positive constant C. By the obvious generalization of Theorem 3.6 to polygonal drawings, we have that the number of crossings is at least

$$\frac{1}{40}C^2n^2 - \frac{1}{16}\sum_{i=1}^{n} 3^2 > \frac{1}{50}C^2n^2,$$

provided that n is large enough. Thus, with probability tending to 1 as $n \to \infty$, we have $L > nC^2/100$. ■

Pach and Wenger [66] also designed a drawing algorithm for arbitrary planar graphs, with the property that every edge is represented by a polygonal path of length $O(n)$.

The proof of the following result is almost identical to the last argument.

Corollary 7.9 *[58]. There exist two point sets $\{p_1, \ldots, p_n\}, \{q_1, \ldots, q_n\}$ in the plane with the property that any piecewise linear homeomorphism from the plane to the itself, for which $f(p_i) = q_i \ \ (i = 1, \ldots, n)$, consists of at least constant times n^2 linear pieces.*

D. Souvaine and R. Wenger [69] showed that there always exists a homeomorphism meeting the requirements of the last corollary, which consists of $O(n^2)$ linear pieces.

Finally, we mention an old application of Theorem 5.2.

Corollary 7.10 *[36]. The maximum number of point pairs realizing the diameter of a n-element set in the plane is equal to n.*

Proof It is enough to notice that connecting two elements of a set P by a segment if their distance is equal to the diameter of P, we obtain a geometric graph with no two disjoint edges. The result follows from Theorem 5.2. ∎

In view of this statement, there is always a point whose degree in the graph of diameters is at most 2. Thus, the chromatic number of this graph is at most 3, and the diameter of each colour class is smaller than 1. This is a very special case of Borsuk's famous conjecture [11], which turned out to be false in high dimensions [38].

Corollary 7.11 *Every finite point set in the plane can be partitioned into three subsets of smaller diameter.*

Z. Füredi [26] applied a Turán-type result with a class of forbidden 5-point geometric subgraphs to prove that the maximum number of times that the same distance can occur among n vertices of a convex polygon is $O(n \log n)$. The best known lower bound, $2n - 7$, is due to Edelsbrunner and Hajnal [19]. Erdős and L. Moser conjectured that the true order of magnitude of this function is $\Theta(n)$.

8 Geometric hypergraphs

It seems plausible that to extend the incidence results to higher dimensions, to improve the upper bound for the number of times the unit distance can occur among n points in 3-space, or to make progress concerning the higher dimensional analogue of the k-set problem, one has to find the right generalizations of the results discussed in the preceding sections to systems of surfaces or surface patches in d-space. For simplicity, we will discuss only the case when these surface patches are flat (simplices).

Definition A *d-dimensional geometric r-hypergraph* H_r^d is a pair (V, E), where V is a set of points in general position in $\mathbb{R}^d$, and E is a set of closed $(r-1)$-dimensional simplices induced by some r-tuples of V. The sets V and E are called the *vertex set* and *edge set* of H_r^d, respectively.

Clearly, a *geometric graph* is a 2-dimensional geometric 2-hypergraph.

Given any class $\mathcal{F}$ of *forbidden* geometric hypergraphs, let $\mathrm{ex}_r^d(\mathcal{F}, n)$ denote the maximum number of edges that a d-dimensional geometric r-hypergraph H_r^d of n vertices can have without containing a geometric subhypergraph which belongs to $\mathcal{F}$. Most bounds in this section are asymptotic: d and r are thought to be fixed, while n tends to infinity.

Akiyama and Alon proved the following theorem.

Theorem 8.1 *[4]. Let $V = V_1 \cup \ldots \cup V_d$ ($|V_1| = \ldots = |V_d| = n$) be a dn-element set in general position in $\mathbb{R}^d$, and let E consist of all $(d-1)$-dimensional simplices having exactly one vertex in each V_i. Then E contains n disjoint simplices.*

According to a well-known result of Erdős [21], any abstract d-uniform hypergraph with n vertices and at least $n^{d-(1/k)^{d-1}}$ hyperedges contains a complete d-partite subhypergraph with k elements in each of its classes. Combining this with the previous result, we obtain

Theorem 8.2 *[4] Let $\mathcal{D}_k^d$ denote the class of all d-dimensional geometric d-hypergraphs consisting of k pairwise disjoint edges. Then the maximum number of hyperedges that a d-dimensional geometric d-hypergraph of n vertices can have without containing k pairwise disjoint hyperedges, satisfies*

$$\mathrm{ex}\left(\mathcal{D}_k^d, n\right) < n^{d-(1/k)^{d-1}}.$$

However, it is conjectured that the true order of magnitude of $\mathrm{ex}\left(\mathcal{D}_k^d, n\right)$ is $O(n^{d-1})$. Theorems 5.3 and 5.4 show that this is the case when $d = 2$.

If we want to exclude *crossings* rather than disjoint edges, or want to generalize Theorem 3.4 to geometric hypergraphs, we face the following problem. Even if we restrict our attention to systems of triangles induced by 3-dimensional point sets in general position, it is not completely clear how a "crossing" should be defined. If two segments cross, they do not share an endpoint. Should this remain true for triangles? We have to clarify the terminology.

Definition Two simplices are said to have a *non-trivial intersection*, if their relative interiors have a point in common. If, in addition, the two simplices are vertex disjoint, then they are said to *cross*.

More generally, k simplices are said to have a *non-trivial intersection*, if their relative interiors have a point in common. If, in addition, all simplices are vertex disjoint, then they are said to *cross*.

Consider k simplices. The fact that *every pair* of them has a non-trivial intersection does not imply that *all* of them do. To emphasize that this stronger condition is satisfied, we often say that the simplices have a *non-trivial intersection in the strong sense*, or simply that they *strongly intersect*. Similarly, a set of *pairwise* crossing simplices is not necessarily *crossing*. If we want to emphasize that they *all* cross, we say that they *cross in the strong sense*, or, in brief, that they *strongly cross*.

Definition Let $\mathcal{I}_k^r$ (respectively, $\mathcal{SI}_k^r$) denote the class of all geometric hypergraphs consisting of k $(r-1)$-dimensional simplices, any two of which have a non-trivial intersection (respectively, all of which are strongly intersecting). Similarly, let $\mathcal{C}_k^r$ (respectively, $\mathcal{SC}_k^r$) denote the class of all geometric hypergraphs consisting of k pairwise crossing (respectively, strongly crossing) $(r-1)$-simplices.

Theorem 8.3 *[8],[18]. For any fixed $k > 1$, one can select at most $O(n^{\lceil d/2\rceil})$ d-dimensional simplices induced by n points in d-space with the property that no k of them share a common interior point. This bound cannot be improved. That is,*

$$\mathrm{ex}_{d+1}^d(\mathcal{I}_k^{d+1}, n) = \Theta(n^{\lceil d/2\rceil}), \qquad \mathrm{ex}_{d+1}^d(\mathcal{SI}_k^{d+1}, n) = \Theta(n^{\lceil d/2\rceil}).$$

Proof Clearly, we have

$$\Omega(n^{\lceil d/2\rceil}) \le \mathrm{ex}_{d+1}^d(\mathcal{I}_k^{d+1}, n) \le \mathrm{ex}_{d+1}^d(\mathcal{SI}_k^{d+1}, n),$$

where the first inequality follows from the fact that there are triangulations of size $\Omega(n^{\lceil d/2\rceil})$ with n vertices in $\mathbb{R}^d$. Consider, for example, the vertical projection of the lower part of any cyclic polytope of n vertices in $\mathbb{R}^{d+1}$.

To see that $\mathrm{ex}_{d+1}^d(\mathcal{SI}_k^{d+1}, n) \le O(n^{\lceil d/2\rceil})$, we set up a charging scheme. Let us regard $\mathbb{R}^{d-1}$ as the coordinate hyperplane in $\mathbb{R}^d$ spanned by the first $d-1$ axes, and let X_d denote the last coordinate axis. Suppose that X_d is vertical. Fix a geometric hypergraph $H_{d+1}^d = (V, E)$ which has no k edges with a common interior point and whose n vertices are in general position. For any l-dimensional simplex Δ induced by V, where $l \le \lfloor (d-1)/2\rfloor$, let $E_\Delta \subseteq E$ denote the set of all edges of H_{d+1}^d that contain Δ on their boundaries. It follows from the condition on H_{d+1}^d that the infinite vertical cylinder $\Delta + X_d$ based on Δ intersects the interior of at most $2(k-1)$ elements of E_Δ. Let us charge Δ one unit for each of these edges. Since the total number of l-simplices with $l \le \lfloor (d-1)/2\rfloor$ is at most $\lceil d/2\rceil n^{\lceil d/2\rceil}$, it remains to show that every edge $e \in E$ has been charged for. Indeed, by Radon's theorem [67], the vertex set of the orthogonal projection of e into $\mathbb{R}^{d-1}$ can be partitioned into two parts, S_1 and S_2, such that their convex hulls cross each other and $|S_1| + |S_2| = d+1$. Suppose without loss of generality that $|S_1| \le \lfloor (d+1)/2\rfloor$. Then the convex hull of S_1 is an l-dimensional simplex Δ_1 for some $l \le \lfloor (d-1)/2\rfloor$, and we had to charge Δ_1 for e.

Theorem 8.4 *[18]. Let E be any set of d-dimensional simplices induced by an n-element point set $V \subseteq \mathbb{R}^d$. If E has no two crossing elements, then $|E| = O(n^d)$, and this bound is asymptotically tight. In notation,*

$$\mathrm{ex}_{d+1}^{d}(\mathcal{C}_2^{d+1}, n) = \Theta(n^d).$$

The last two theorems were about d-dimensional geometric $d+1$-hypergraphs, i.e., systems of full-dimensional simplices induced by a set of points.

Next we present a few statements on d-dimensional geometric d-hypergraphs. They can be viewed as natural extensions of the corresponding results for geometric graphs (the case $d = 2$) discussed in the preceding sections.

Theorem 8.5 *[18],[79]. Let E be a family of $(d-1)$-dimensional simplices induced by an n-element point set $V \subseteq \mathbb{R}^d$ such that E has no k members with pairwise non-trivial intersections $(d, k > 1)$. Then, for $k = 2$ and 3, we have $|E| = O(n^{d-1})$. Otherwise, $|E| = O(n^{d-1}\log n)$. In notation,*

$$\mathrm{ex}_d^d(\mathcal{I}_k^d, n) = O(n^{d-1}) \quad \textit{if } k = 2, 3;$$

$$\mathrm{ex}_d^d(\mathcal{I}_k^d, n) = O(n^{d-1}\log n) \quad \textit{otherwise.}$$

This result is asymptotically tight if $d, k \leq 3$.

Proof For $d = 2$, the assertion is true, by Theorem 4.1 and by Corollary 4.4. Assume that $d \geq 3$. For any $(d-3)$-simplex Δ induced by V, let E_Δ denote the family of all members of E that contain Δ as a face. Pick any point p_Δ in the relative interior of Δ, and let F_Δ denote the 3-dimensional flat orthogonal to Δ and passing through p_Δ.

Every $e \in E_\Delta$ meets F_Δ in a polygon, whose two sides incident to p_Δ are the intersections of F_Δ with the two $(d-2)$-faces of e containing Δ. Thus, the total number of sides incident to p_Δ that occur in some $e \cap F_\Delta$ $(e \in E_\Delta)$ is at most $n - d + 2 < n$. Take a small 2-dimensional sphere $S^2 \subseteq F_\Delta$ centered at p_Δ. The intersections of S^2 with the elements of E_Δ form the edge set of a graph with at most n vertices. It follows from the properties of E that this graph has no k pairwise crossing edges, so, by the planar results, its number of edges, $|E_\Delta|$, satisfies

$$|E_\Delta| = O(n) \quad \text{if } k = 2, 3;$$

$$|E_\Delta| = O(n \log n) \quad \text{otherwise.}$$

Summing over all $(d-3)$-simplices Δ induced by V, we obtain $\binom{d}{2}|E| = \sum_\Delta |E_\Delta|$, and hence the upper bound.

To show that the result is tight for $d = 3, k = 2$, consider a nested sequence of $n/2$ pyramids based on the same 2-dimensional convex $n/2$-gon. These pyramids have a total of $n^2/4$ triangular faces, no two of which have a non-trivial intersection. ■

It is an outstanding open problem to decide whether the order of magnitude of the above bound can be improved, e.g., for $d=4, k=2$. However, we have the following related result.

Theorem 8.6 *[18]. Let E be a family of $(d-1)$-dimensional simplices induced by an n-element point set $V \subseteq \mathbb{R}^d$. If E has no two crossing members, then $|E| = O(n^{d-1})$, and this bound cannot be improved. In notation,*

$$\mathrm{ex}_d^d(\mathcal{C}_2^d, n) = \Theta(n^{d-1}).$$

This can be generalized as follows.

Theorem 8.7 *Let E be a family of $(d-1)$-dimensional simplices induced by an n-element point set $V \subseteq \mathbb{R}^d$, where $d, k > 1$. If E has no k pairwise crossing members, then $|E| = O(n^{d-(1/d)^{k-2}})$. In notation,*

$$\mathrm{ex}_d^d(\mathcal{C}_k^d, n) = O(n^{d-(1/d)^{k-2}}).$$

The proof of this result is based on a natural extension of Theorem 4.8, which provides us with a recipe for giving a lower bound on the *number* of crossing k-tuples of edges, if we know how many edges are necessary to guarantee the existence of *one* such k-tuple.

Theorem 8.8 *[18]. Assume that, for suitable constants c_1 and $0 \le \delta \le 1$, we have $\mathrm{ex}_r^d(\mathcal{SC}_k^r, n) < c_1\binom{n}{r}/n^{\delta}$ and $e \ge (c_1+1)\binom{n}{r}/n^{\delta}$.*

Then there exists $c_2 > 0$ such that the minimum number of strongly crossing k-tuples of edges in a d-dimensional r-hypergraph with n vertices and e edges is at least

$$c_2\binom{n}{kr}e^{\gamma}/\binom{n}{r}^{\gamma},$$

where $\gamma = 1 + \frac{(k-1)r}{\delta}$.

As Theorem 3.4 can be applied to the planar k-set problem (compare Theorem 7.2), the last result, which is a far-reaching generalization of Theorem 3.4, is relevant to its higher dimensional analogues.

Given a set P of n points in 3-space, in general position, a k-element subset $P' \subseteq P$ is called a k-set, if there is a half-space H such that $H \cap P = P'$. Bárány, Füredi and Lovász [9] gave the first non-trivial (subcubic) upper bound for the number of $\lfloor n/2 \rfloor$-sets of an n-element set in 3-space. The best known bound is given by the following theorem.

Theorem 8.9 *[17], [6]. The number of $\lfloor n/2 \rfloor$-sets of an n-element set in general position in 3-space is $O(n^{8/3})$.*

Proof Let P be a set of n points in general position in 3-space. Similarly to the 2-dimensional case, it is sufficient to bound the number of edges in the 3-dimensional geometric 3-hypergraph $H = (V, E)$, where $V = P$ and E consists of all triangles whose supporting planes pass through 3 elements of P, and which have precisely $\lfloor n/2 \rfloor$ points on one of their sides. Let $e := |E|$.

In view of Theorem 8.6, we can apply Theorem 8.8 with $d = r = 3,\ k = 2$, and $\delta = 1$ to obtain that the number of (strongly) crossing pairs of edges of H is at least $\Omega(e^4/n^6)$, provided that $e > cn^2$ for a suitable constant $c > 0$. Notice that if two triangles cross each other, then one of them must have an edge that crosses the other triangle. Thus, there is an edge crossing at least $\Omega(e^3/n^6)$ triangles. On the other hand, by an old (and easy) observation of Lovász, every straight line crosses at most n^2 edges of H. We have

$$n^2 = \Omega(e^3/n^6),$$

and the result follows. ■

We close this section with a generalization of Theorem 6.2.

Theorem 8.10 *[18]. Let us colour with two colours all $(d-1)$-dimensional simplices induced by $(d+1)n - 1$ points in general position in $\mathbb{R}^d$. Then one can always find n disjoint simplices of the same colour. This result cannot be improved.*

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