Bichromatic lines with few points

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Abstract

Given a set of n blue and n red points in the plane, not all on a line, it is shown that there exists a bichromatic line passing through at most two blue points and at most two red points. There does not necessarily exist a line passing through precisely one blue and one red point. This result is extended to the case when the number of blue and red points are not the same.

1 Introduction

According to a celebrated result of Sylvester [S93] and Gallai [G44], any set P of non-collinear points in the plane determines a so-called *ordinary* line, i.e., a line passing through precisely two elements of P. Recently, Fukuda [F96, DSF98] conjectured that this result can be generalized as follows. Let R be a set of red points and let B be a set of blue points in the plane, not all on a straight line. Assume that

(i) R and B are separated by a straight line, and
(ii) $|R|$ and $|B|$ differ by at most one.

Then there exists a *bichromatic* ordinary line, i.e., a line passing through precisely one red and one blue point. It is not hard to see that this statement does not remain true if we drop any of the above assumptions. (See Remarks 4.1-2.) The aim of the present note is to show that without making any special assumption it is still true that there always exist bichromatic lines containing relatively few points of $R \cup B$.

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Theorem 1. *Given n blue points and $N = cn$ $(c \geq 1)$ red points in the plane, not all on a line, the number of bichromatic lines passing through at most $8c$ points is at least $1/(25c^2)$ times the total number of connecting lines. In particular, there exists at least one such line.*

In the special case $N = n$, we establish a somewhat stronger result.

Theorem 2. *Given n red and n blue points in the plane, not all on a line,*
(i) there exist more than $n/2$ bichromatic lines that pass through at most two red points and at most two blue points;
(ii) the number of bichromatic lines passing through at most six points is at least one tenth of the total number of connecting lines.

The proofs are based on the following simple consequence of Euler's Polyhedral Formula (see e.g. [AZ98]), which immediately implies the Sylvester-Gallai theorem.

Lemma. *Let P be a finite non-collinear point set in the plane, and let l_i $(i = 2, 3, \ldots)$ denote the number of lines passing through precisely i elements of P. Then we have*

$$\sum_{i=2}^{n-1} (i-3) l_i \leq -3. \quad \square$$

We start with the proof of Theorem 2, because it uses similar tricks but requires less tedious computations than the proof of Theorem 1. The proof of Theorems 2 and 1 are presented in Sections 2 and 3, respectively, while the last section contains some constructions and remarks.

2 Proof of Theorem 2

Let R and B be two disjoint n-element point sets in the plane, and assume that not all elements of $R \cup B$ are on the same line. We will refer to the elements of R and B, as *red* points and *blue* points, respectively.

For any ordered pair of non-negative integers (i, j), $\;i + j \geq 2$, let l_{ij} denote the number of lines passing through precisely i red and j blue points. In particular, the number of bichromatic lines is $\sum_{i,j\geq 1} l_{ij}$. Set $l_{ij} := 0$, whenever $i + j \leq 1$.

The number of monochromatic point pairs is equal to

$$\sum_{i,j\geq 0} \left[\binom{i}{2} + \binom{j}{2} \right] l_{ij} = 2\binom{n}{2} = n^2 - n.$$

The number of bichromatic point pairs is $\sum_{i,j\geq 0} ijl_{ij} = n^2$. Thus, we have

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} \left[\binom{i}{2} + \binom{j}{2} - ij\right] l_{ij} = -n. \tag{1}$$

The Lemma at the end of the previous section implies that

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} (i+j-3)l_{ij} \leq -3. \tag{2}$$

Adding up twice (1) and $1+\varepsilon$ times (2), for some positive ε, we obtain

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} \left[(i-j)^2 + \varepsilon(i+j-3) - 3\right] l_{ij} \leq -2n - 3(1+\varepsilon). \tag{3}$$

For any $i,j \geq 0$ ($i+j\geq 2$), let γ_{ij} denote the coefficient of l_{ij} in the above inequality, so that $\sum_{i,j} \gamma_{ij} l_{ij} < 0$.

First, set $\varepsilon = 1$. It is easy to verify that $\gamma_{11} = -4$, $\gamma_{12} = \gamma_{21} = \gamma_{22} = -2$, and that all other coefficients γ_{ij} are non-negative. Therefore, (3) yields that

$$-4l_{11} - 2l_{12} - 2l_{21} - 2l_{22} \leq -2n - 6.$$

Consequently,

$$2l_{11} + l_{12} + l_{21} + l_{22} \geq n + 3,$$

which proves part (i) of Theorem 2.

To establish part (ii), set $\varepsilon = 3/5$. Then

$$\gamma_{11} = -\frac{18}{5},\ \gamma_{12} = \gamma_{21} = -2,\ \gamma_{22} = -\frac{12}{5},\ \gamma_{23} = \gamma_{32} = -\frac{4}{5},\ \gamma_{33} = -\frac{6}{5},$$

and all other coefficients are at least $2/5$. Hence,

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}} \gamma_{ij}l_{ij} + 4l_{11} + \frac{12}{5}l_{12} + \frac{12}{5}l_{21} + \frac{14}{5}l_{22} + \frac{6}{5}l_{23} + \frac{6}{5}l_{32} + \frac{8}{5}l_{33} \geq \frac{2}{5} \cdot \sum_{\substack{i,j\geq 0\\ i+j\geq 2}} l_{ij}.$$

Comparing the last inequality with (3), we obtain

$$-2n - 3\left(1+\frac{3}{5}\right) + 4l_{11} + \frac{12}{5}l_{12} + \frac{12}{5}l_{21} + \frac{14}{5}l_{22} + \frac{6}{5}l_{23} + \frac{6}{5}l_{32} + \frac{8}{5}l_{33} \geq \frac{2}{5} \cdot \sum_{\substack{i,j\geq 0\\ i+j\geq 2}} l_{ij}.$$

That is,

$$\sum_{\substack{i,j\geq 0\\ 2\leq i+j\leq 6}} l_{ij} \geq l_{11}+\frac{3}{5}l_{12}+\frac{3}{5}l_{21}+\frac{7}{10}l_{22}+\frac{3}{10}l_{23}+\frac{3}{10}l_{32}+\frac{2}{5}l_{33}$$

$$\geq \frac{1}{10}\cdot\sum_{i,j\geq 0} l_{ij}+\frac{n}{2}+\frac{6}{5},$$

which completes the proof of part (ii) of the theorem.

3 Proof of Theorem 1

Throughout this section, let R and B denote fixed disjoint sets of N red and $n \leq N$ blue points in the plane, respectively. Assume that not all elements of $R \cup B$ are on the same line, and let $c := N/n \geq 1$.

For any non-negative integers i, j, $(i+j \geq 2)$, let l_{ij} denote the number of lines passing through precisely i red and j blue points. Let $l_{ij} := 0$, whenever $i + j \leq 1$.

Assign a weight w to every unordered pair of points $\{p, q\} \in R \cup B$, as follows. Let

$$w(\{p,q\}) = \begin{cases} 1 & \text{if } p \text{ and } q \text{ are red,}\\ c^2 & \text{if } p \text{ and } q \text{ are blue,}\\ -c & \text{otherwise.}\end{cases}$$

Clearly, the total weight of monochromatic pairs is

$$\binom{N}{2}+c^2\binom{n}{2}<\frac{N^2}{2}+\frac{c^2n^2}{2}.$$

Writing the same quantity differently, we obtain

$$\sum_{i,j\geq 0}\left[\binom{i}{2}+c^2\binom{j}{2}\right]l_{ij}<\frac{N^2}{2}+\frac{c^2n^2}{2}.$$

The total weight of bichromatic pairs is

$$-c\sum_{i,j\geq 0} ijl_{ij}=-cNn.$$

Summing up the last two relations, and multiplying by 2, we get

$$\sum_{i,j\geq 0}\left[(i-cj)^2-i-c^2j\right]l_{ij}<(N-cn)^2=0. \tag{4}$$

Suppose first that B is not collinear. It follows from the Lemma in Section 1, applied to B and $R\cup B$, respectively, that

$$\sum_{j\geq 2}(j-3)\sum_{i\geq 0}l_{ij}<0, \tag{5}$$

$$\sum_{\substack{i,j\geq 0\\ i+j\geq 2}}(i+j-3)l_{ij}<0. \tag{6}$$

Let $0<\varepsilon<1$ be a constant to be specified later. Adding up (4), $2c^2-2$ times (5) and $2-\varepsilon$ times (6), we obtain

$$\begin{aligned}&\sum_{\substack{i\geq 2\\ j=0,1}}\left[i^2+i-2cij-\varepsilon i+(2-\varepsilon)(j-3)\right]l_{ij}-(2c-2+\varepsilon)l_{11}\\ &+\sum_{\substack{i\geq 0\\ j\geq 2}}\left[(i-cj)^2+(1-\varepsilon)i+(c^2-\varepsilon)j-3(2c^2-\varepsilon)\right]l_{ij}<0.\end{aligned} \tag{7}$$

Let γ_{ij} denote the coefficient of l_{ij} in (7). It is easy to verify that these numbers satisfy the following conditions.

1. $\gamma_{i0},\gamma_{0j}\geq\varepsilon$ for every $i\geq 2, j\geq 2$.
2. $\gamma_{i1}>\varepsilon$ for every $i\geq 2c+2$.
3. $\gamma_{i1}\geq-\left(c+\frac{\varepsilon-1}{2}\right)^2+2\varepsilon-4$ for every $1\leq i<2c+2$.
4. Keeping $k=i+j$ fixed, γ_{ij} attains its minimum when $j=j_0=\frac{2k-c+1}{2c+2}$. We have

   $$\gamma_{k-j_0,j_0}=(c-\varepsilon)k-\frac{(c-1)^2}{4}-6c^2+3\varepsilon.$$

   Therefore, for any $i\geq 1, j\geq 2$,

   $$\gamma_{ij}\geq(c-\varepsilon)(i+j)-\frac{(c-1)^2}{4}-6c^2+3\varepsilon\geq\varepsilon,$$

provided that $i+j$ is at least

$$k_0 = \frac{25c^2 - 2c + 1 - 8\varepsilon}{4(c-\varepsilon)}. \tag{8}$$

5. On the other hand, if $i \geq 1, j \geq 2$ and $i+j < k_0$, it follows that

$$\gamma_{ij} \geq (c-\varepsilon)3 - \frac{(c-1)^2}{4} - 6c^2 + 3\varepsilon = -\frac{25c^2 - 14c + 1}{4}.$$

Plugging these conditions into (7), we obtain

$$\varepsilon \cdot \left[\sum_{\substack{i=0\\ j\geq 2}} l_{ij} + \sum_{\substack{i\geq 2\\ j=0}} l_{ij} + \sum_{\substack{i\geq 2c+2\\ j=1}} l_{ij} + \sum_{\substack{i\geq 1, j\geq 2\\ i+j\geq k_0}} l_{ij} \right]$$

$$- \left[\left(c + \frac{\varepsilon - 1}{2}\right)^2 - 2\varepsilon + 4 \right] \cdot \sum_{\substack{1\leq i<2c+2\\ j=1}} l_{ij} - \frac{25c^2 - 14c + 1}{4} \cdot \sum_{\substack{i\geq 1, j\geq 2\\ i+j<k_0}} l_{ij} < 0.$$

Equivalently,

$$\left[\left(c + \frac{\varepsilon - 1}{2}\right)^2 - \varepsilon + 4 \right] \cdot \sum_{\substack{1\leq i<2c+2\\ j=1}} l_{ij}$$

$$+ \left[\frac{25c^2 - 14c + 1}{4} + \varepsilon \right] \cdot \sum_{\substack{i\geq 1, j\geq 2\\ i+j<k_0}} l_{ij} > \varepsilon \cdot \sum_{\substack{i\geq 0\\ j\geq 0}} l_{ij}.$$

It can be shown by straightforward computation that both coefficients on the left-hand side of the last inequality can be estimated from above by $25c^2/4$, so that

$$\sum_{\substack{1\leq i<2c+2\\ j=1}} l_{ij} + \sum_{\substack{i\geq 1, j\geq 2\\ i+j<k_0}} l_{ij} > \frac{4\varepsilon}{25c^2} \cdot \sum_{\substack{i\geq 0\\ j\geq 0}} l_{ij}.$$

Setting $\varepsilon = 1/4$, say, in view of (8), the result follows.

It remains to verify that Theorem 1 is also valid in the case when all points of B are on a line ℓ, i.e., when (5) does not hold. Let s denote the number of points of R which belong to ℓ. According to our assumption, $R \cup B$ is not collinear, so we have $s < |R| = cn$. For any $p \in B$, let $l(p)$ denote the number of bichromatic lines through p, different from ℓ, that contain at most $2c$ red points. There are at least $cn - s - 2cl(p)$ red points not belonging to ℓ, whose connecting lines with p pass through more than $2c$ red points. Thus, there are at least

$$\frac{cn - s - 2cl(p)}{2c+1} \cdot \binom{2c+1}{2}$$

pairs of red points not on ℓ such that the line induced by them passes through p. Since any such line can pass through at most one point of B, we obtain

$$\sum_{p \in B} \frac{cn - s - 2cl(p)}{2c+1} \cdot \binom{2c+1}{2} \leq \binom{cn-s}{2},$$

whence

$$\sum_{p \in B} l(p) \geq \frac{c^2n^2 - s^2}{4c^2}.$$

The total number L of connecting lines of $R \cup B$ is at most

$$\binom{cn-s}{2} + (cn - s)(n + s) + 1.$$

Hence, the proportion of those bichromatic connecting lines which contain one blue point and at most $2c$ red points is

$$\frac{\sum_{p \in B} l(p)}{L} \geq \frac{1}{8c^2}.$$

This completes the proof of Theorem 1.

4 Concluding remarks

Recall Fukuda's conjecture mentioned in the first paragraph of the Introduction. It states that if R is a set of red points and B is a set of blue points in the plane, not all on a line, and conditions (i) and (ii) are satisfied, then there is a straight line passing through precisely one red and one blue point. In this section, we show that the above assumptions are necessary.

4.1. First we prove that Fukuda's conjecture is false if we drop condition (i).

Suppose that n is even, say, $n = 2k$. Let P be the vertex set of a regular n-gon, and let Q be the set of intersection points of the line at infinity with all lines determined by two elements of P. Clearly, we have $|P| = |Q| = n$. Color $P \cup Q$ with two colors, red and blue, so that the number of red points equals the number of blue points and every *ordinary line* (i.e., every line passing through precisely two points) is *monochromatic.*

Notice that every ordinary line determined by $P \cup Q$ passes through one element of Q. Furthermore, for every pair of opposite vertices $p_1, p_2 \in P$, there is a unique point $q \in Q$, such that p_1q and p_2q are ordinary lines. Pick $\lfloor \frac{k}{2} \rfloor$ pairs of opposite vertices of P and the corresponding points on the line at infinity, and color them red. Color the remaining $\lceil \frac{k}{2} \rceil$ pairs of opposite vertices of P and the corresponding points on the line at infinity blue. Finally, color the k uncolored points of Q so that to balance the number of red and blue points. Obviously, all ordinary lines determined by $P \cup Q$ are monochromatic.

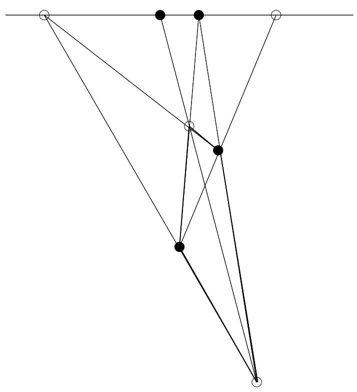

Figure 1: R and B are not separated, and there is no bichromatic ordinary line

If we wish to avoid using points at infinity, we can modify this construction by applying a suitable projective transformation (see Figure 1). Similar constructions can be given in the case when $n > 3$ is odd.

Our constructions also imply that in a certain sense part (i) of Theorem 2 is best possible. However, it is possible that even without assuming (i) there always exists a bichromatic line passing through at most *three* points.

4.2. As is illustrated by the 6-element set depicted in Figure 2, Fukuda's conjecture does not remain true if we drop condition (ii) guaranteeing that the number of red points differ and the number of blue points fiffer by at most *one*.

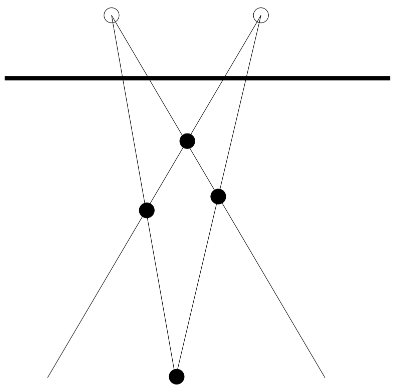

Figure 2: Unbalanced set of 6 points with no bichromatic ordinary line

To obtain some larger examples, let $n = 2k$ and let R denote a regular n-gon. Let B consist of all intersection points of the line at infinity with the lines induced by the sides of R. Clearly, we have $|B| = n/2 = k$. Coloring all points o R and B with red and blue, respectively, we obtain an unbalanced example with twice as many red points as blue points such that they are separated by a straight line, but there exists no monochromatic ordinary line. Again, if we wish, we can apply a suitable projective transformation to get rid of all points at infinity (see Figure 3).

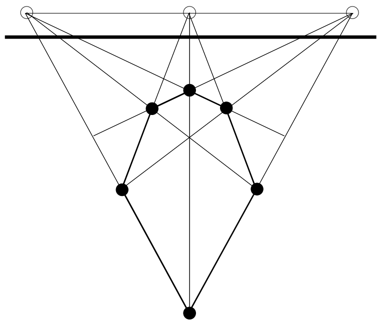

Figure 3: Twice as many red points as blue points, separated by a line

4.3. One may think that the following statement, which can be regarded as a dual counterpart of Fukuda's conjecture, is also true. Let R and B non-collinear point sets in the plane satisfying conditions (i) and (ii) in the first paragraph of the Introduction. Then there is a *monochromatic* ordinary line, i.e., a line passing through precisely two points, which have the same color.

However, the above statement is false. Let k be a positive integer, and let

$$R = \{\ (1,i)\ :\ |i| \leq k\ \} \cup \{(\infty,0)\},$$
$$B = \{\ (-1,i)\ :\ |i| \leq k\ \} \cup \{(0,0)\},$$

where $(\infty,0)$ denoted the intersection point of the x-axis with the line at infinity. Obviously, $R \cup B$ determines no monochromatic ordinary line. If we apply a suitable projective transformation we get rid of the point at infinity (see Figure 4).

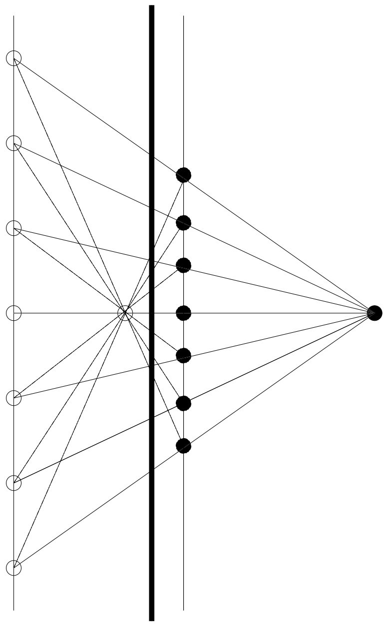

Figure 4: Balanced and separated set with no monochromatic ordinary line

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