On The Number Of Balanced Lines

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Abstract

Given a set of n black and n white points in general position in the plane, a line l determined by them is said to be *balanced* if each open half-plane bounded by l contains precisely the same number of black points as white points. It is proved that the number of balanced lines is at least n. This settles a conjecture of George Baloglou.

1 Introduction

Throughout this paper, let V be a set of $2n$ points in general position in the plane, i.e., assume that no three of them are on a line. Suppose that half of the points have weight $+1$ and the other half weight -1. We say that a line passing through two elements of V is *determined* by V.

Definition 1.1. *A line l determined by V is called* balanced *if in each open half-plane bounded by l the total weight of the points is* 0*.*

The following observation is an immediate consequence of the definition.

Claim 1.2. *If two points determine a balanced line l, then they have opposite weights.*

Indeed, since the total weight of the points as well as the total weight of all points *not on* l is 0, it follows that the sum of the weights of the two points *on* l must be 0, too.

In view of the claim, the number of balanced lines determined by V cannot exceed n^2. This bound is attained by many configurations, including every convex $2n$-gon whose vertices are of weight $+1$ and -1, alternately.

The aim of this paper is to prove the following conjecture of George Baloglou.

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Theorem 1.3. *Every set V consisting of n points of weight $+1$ and n points of weight -1 in general position in the plane determines at least n balanced lines. This bound cannot be improved.*

The tightness of the above theorem is shown e.g. by a convex $2n$-gon, whose vertices of weight $+1$ are separated from the vertices of weight -1 by a straight line. In fact, we have

Theorem 1.4. *Let V be a set of $2n$ points in general position in the plane, consisting of n points of weight $+1$ and n points of weight -1 separated by a straight line.*

Then V determines precisely n balanced lines.

It is sufficient to prove Theorem 1.3 in the special case when no two lines determined by V are parallel, and in the sequel we assume that V satisfies this condition.

It is easy to verify

Claim 1.5. *For any vertex v of the convex hull of V, there is a balanced line passing through v.*

Proof: Let $u_1, \dots, u_{2n-1}$ denote the elements of $V \setminus \{v\}$ listed in clockwise order of visibility from v. Suppose without loss of generality that the weight of v is positive. If u_1 or u_{2n-1} has negative weight, then we are done, because in this case vu_1 resp. vu_{2n-1} is a balanced line. Take the line vu_1, start rotating it clockwise around v, and keep track of the total weight L of the elements of V in the open half-plane to the left of this line. At the moment when the line passes through u_2, we have $L = 1$. Finally, the line passes through u_{2n-1} and $L = -2$. Every time the line passes through a new point the value of L changes by 1, so there is a maximum index $i > 2$ such that the total weight of the points on the left-hand side of vu_i is 0. By the maximality of i, the weight of u_i must be negative. Therefore, the total weight of the points on the right-hand side of vu_i is also 0, i.e., vu_i is a balanced line. ∎

It may be tempting to believe that Claim 1.5 is also true for all points of V lying in the *interior* of the convex hull of V, which would immediately imply Theorem 1.3. However, as is illustrated by Figure 1, this is not necessarily the case.

For the proof of Theorem 1.3, we need the notion of a *flip array* associated with V. (In the literature it is often called a *circular sequence* or an *allowable sequence* of permutations [GP93].)

Fix an orthogonal coordinate system (x, y) in the plane so that no two elements of V have the same x-coordinate. Let $v_1, \dots, v_{2n}$ denote the elements of V in increasing order of their x-coordinates. For notational convenience, in the sequel we identify v_i with i, and we write $w(i)$ for the weight of v_i. The flip array associated with V is a sequence of $\binom{2n}{2} + 1$ permutations of the elements $1, \dots, 2n$, denoted by P_t $\left(0 \leq t \leq \binom{2n}{2}\right)$. Start

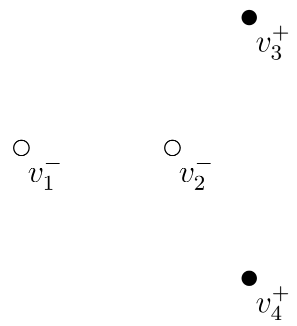

Figure 1: v_2 is not incident to any balanced line

rotating a directed line l parallel to the x-axis in the clockwise direction, and consider the permutations determined by the order, in which the elements of V fall on l. Originally, this order is $P_0 = (1, \ldots, 2n)$. Suppose that we have already defined the permutations $P_0, \ldots, P_{t-1}$ for some $t \leq \binom{2n}{2}$, and continue rotating l. A new permutation arises whenever l passes through a direction orthogonal to a line l_t determined by two points $v_i, v_j \in V$. Then i and j are consecutive elements in P_{t-1}, and P_t can be obtained from P_{t-1} by reversing their order. Such a move is called a *flip* or a *swap*. After rotating l through a half turn π, we obtain $P_{\binom{2n}{2}} = (2n, 2n-1, \ldots, 1)$, and then we stop. For any $0 \leq t \leq \binom{2n}{2}$ and $1 \leq i \leq 2n$, let $p_{t,i}$ denote the i-th element of P_t. That is, we have $P_t = (p_{t,1}, \ldots, p_{t,2n})$.

We have to introduce some further notations.

Definition 1.6. *For any $0 \leq t \leq \binom{2n}{2}$ and $1 \leq i \leq 2n$, let $L_t(i)$ denote the sum of the weights of the first $i-1$ elements of P_t. In other words, let*

$$L_t(i) := \sum_{1 \leq j < i} w(p_{t,j}).$$

Similarly, let

$$R_t(i) := \sum_{i < j \leq 2n} w(p_{t,j}).$$

Definition 1.7. *For every $S \subseteq \{1, 2, \ldots, 2n\}$ and $0 \leq t \leq \binom{2n}{2}$, let $S^L_{t,1} < S^L_{t,2} < \ldots < S^L_{t,|S|}$ denote the positions in P_t occupied by the elements of S, listed from left to right. In other words, $S^L_{t,i}$ denotes the position of the i-th leftmost element of S in P_t. Similarly, let $S^R_{t,i}$ denote the position of the i-th rightmost element of S in P_t. Clearly, we have $S^R_{t,i} = S^L_{t,|S|-i+1}$.*

In our notations, the letters L and R stand for Left and Right, respectively.

2 A standard way to obtain balanced lines

Let $A = \{a_1, \ldots, a_n\} \subset \{1, \ldots, 2n\}$ denote the set of all elements of weight $+1$, listed in increasing order.

Call a set $F \subset A$ *prefix* if $F = \{a_1, a_2, \ldots, a_{|F|}\}$. Similarly, $H \subset A$ is said to be a *suffix* set if $H = \{a_{n-|H|+1}, a_{n-|H|+2}, \ldots, a_n\}$.

We present a "standard" method for finding a balanced line passing through an element of a prefix (suffix) set.

Lemma 2.1. *Let F be a prefix set and let $1 \leq t \leq \binom{2n}{2}$, and let l_t denote the line induced by the two points flipped as we pass from P_{t-1} to P_t.*

Whenever we have $L_{t-1}(F^L_{t-1,k}) \geq 0$ and $L_t(F^L_{t,k}) < 0$, then l_t is a balanced line which passes through a point of F, and there are exactly $k-1$ points of F in the open half-plane to the left of l_t.

Proof: Let x denote the element at position $F^L_{t-1,k}$ in P_{t-1}. Observe that x must swap places with some other element, y, when going from P_{t-1} to P_t, for otherwise we would have $L_{t-1}(F^L_{t-1,k}) = L_t(F^L_{t,k})$.

Suppose $y \in F$. Then the elements of F occupy the same positions in P_t as they do in P_{t-1}, except that their internal order is different. Moreover, every element, not in F, remains at the same place in P_t where it was in P_{t-1}. Thus, we would have $F^L_{t,k} = F^L_{t-1,k}$ and $L_{t-1}(F^L_{t-1,k}) = L_t(F^L_{t,k})$, a contradiction. Therefore, we may assume that $y \notin F$.

Assume first that $w(y) = +1$. Since $y \notin F$ and F is prefix, $y > x$. Therefore, in P_{t-1}, y is at the position $F^L_{t-1,k} + 1$. In P_t, x is still the k-th leftmost element of F, and we have $L_t(F^L_{t,k}) = L_{t-1}(F^L_{t-1,k}) + w(y) = L_{t-1}(F^L_{t-1,k}) + 1$, contradicting the assumptions in the lemma.

We are, therefore, left with the case when $w(y) = -1$. If y is at the position $F^L_{t-1,k} - 1$ in P_{t-1}, then $L_t(F^L_{t,k}) = L_{t-1}(F^L_{t-1,k}) - w(y) = L_{t-1}(F^L_{t-1,k}) + 1$, and again we reach a contradiction.

We conclude that y is at position $F^L_{t-1,k}+1$ in P_{t-1}. Therefore, $L_t(F^L_{t,k}) = L_{t-1}(F^L_{t-1,k}) + w(y) = L_{t-1}(F^L_{t-1,k}) - 1$. It follows from the assumption $L_t(F^L_{t,k}) < 0$ and $L_{t-1}(F^L_{t-1,k}) \geq 0$, that $L_{t-1}(F^L_{t-1,k}) = 0$. In other words, the sum of the weights of the points lying in the open half-plane to the left of l_t is 0. Since l_t is determined by two points of opposite weights, it follows that l_t is a balanced line. By the definition of $F^L_{t-1,k}$, the line l_t (which passes through x) has exactly $k-1$ points of F in the open half-plane to its left. ∎

Similarly, we have

Lemma 2.2. *Let H be a suffix set and let $1 \leq t \leq \binom{2n}{2}$, and let l_t denote the line induced by the two points flipped as we pass from P_{t-1} to P_t.*

Whenever we have $R_{t-1}(H^R_{t-1,k}) \geq 0$ and $R_t(H^R_{t,k}) < 0$, then l_t is a balanced line which passes through a point of H, and there are exactly $k-1$ points of H in the open half-plane to the right of l_t.

Before turning to the proof of Theorem 1.3, we establish Theorem 1.4.

Proof of Theorem 1.4: Since the points of weight $+1$ and -1 are separated by a line, by a proper choice of the x-axis, we can attain that in the flip array of V the set of points of positive weight is $F = \{1, 2, \ldots, n\}$. Clearly, F is a prefix set. Using the fact that P_0 is the identity permutation, i.e., $P_0 = (1, 2, \ldots, 2n)$, we obtain that for every $1 \leq i \leq |F| = n$, $F^L_{0,i} = i$ and $L_0(F^L_{0,i}) = i - 1 \geq 0$.

On the other hand, $P_{\binom{2n}{2}} = (2n, 2n-1, \ldots, 2, 1)$. Thus, for every $1 \leq i \leq |F| = n$, $F^L_{\binom{2n}{2},i} = n + i$ and $L_{\binom{2n}{2}}(F^L_{\binom{2n}{2},i}) = -n - 1 + i < 0$.

Fix $1 \leq k \leq n$. $F^L_{t,k}$ is a continuous function of t, i.e., for every $0 < t \leq \binom{2n}{2}$, we have $|F^L_{t,k} - F^L_{t-1,k}| \leq 1$. We claim that $0 \leq L_{t-1}(F^L_{t-1,k}) - L_t(F^L_{t,k}) \leq 1$, whenever $1 \leq t \leq n$. That is, $L_t(F^L_{t,k})$ is a monotone non-increasing, continuous function of t.

Let $x \in F$ denote the element at position $F^L_{t-1,k}$ in P_{t-1}, that is, x is the k-th leftmost element of F in P_{t-1}. If l_t does not pass through x, then x remains the k-th leftmost element of F in P_t, and every element to the left (right) of x in P_{t-1} is to the left (right) of x in P_t. Therefore, we have $L_t(F^L_{t,k}) = L_{t-1}(F^L_{t-1,k})$.

Assume that l_t passes through x. In other words, x changes places with another element y, when going from P_{t-1} to P_t. There are two possibilities:

Case 1. : $y \in F$.

In this case, the elements of F occupy the same positions in P_t as in P_{t-1}, except that their internal order is different. Hence, $F^L_{t,k} = F^L_{t-1,k}$ and $L_t(F^L_{t,k}) = L_{t-1}(F^L_{t-1,k})$.

Case 2. : $y \notin F$.

Now y has weight -1. Since x and y are flipped when we pass from P_{t-1} to P_t, the point y is either at position $F^L_{t-1,k} - 1$ or at position $F^L_{t-1,k} + 1$ in P_{t-1}. The former possibility cannot occur, for if y were at position $F^L_{t-1,k} - 1$ in P_{t-1}, then x and y would have been flipped earlier, which is impossible. Thus, we can assume that y is at position $F^L_{t-1,k} + 1$ in P_{t-1}. Since $y \notin F$ and x is the k-th leftmost element of F in P_{t-1}, we obtain that x remains the k-th leftmost element of F in P_t and $F^L_{t,k} = F^L_{t-1,k} + 1$. Furthermore, we have $L_t(F^L_{t,k}) = L_{t-1}(F^L_{t-1,k}) + w(y) = L_{t-1}(F^L_{t-1,k}) - 1$.

This proves the claim that $L_t(F^L_{t,k})$ is monotone non-increasing, continuous function of t. Since $L_0(F^L_{0,k}) \geq 0$ and $L_{\binom{2n}{2}}(F^L_{\binom{2n}{2},k}) < 0$, it follows that there is a unique $0 < t_k \leq \binom{2n}{2}$ such that $L_{t_k-1}(F^L_{t_k-1,k}) \geq 0$ and $L_{t_k}(F^L_{t_k,k}) < 0$. By Lemma 2.1, l_{t_k} is a balanced line through an element of F, which has exactly $k-1$ elements of F in the open half-plane to its left. Obviously, $l_{t_1}, \ldots, l_{t_n}$ are distinct balanced lines. Next we show that if l_t is a balanced line, then t is one of $t_1, \ldots, t_n$. By Claim 1.2, l_t passes through an element x with weight $+1$ and an element y with weight -1. Suppose that x is the k-th leftmost element of F in P_{t-1} $(1 \le k \le n)$. Then x is at position $F^L_{t-1,k}$ in P_{t-1}. Since $w(y) = -1$, we have $x < y$. Therefore, y is at position $F^L_{t-1,k}+1$ in P_{t-1}. Since l_t is a balanced line, it follows that $L_{t-1}(F^L_{t-1,k}) = 0$. In P_t, x is still the k-th leftmost element of F, and we have $L_t(F^L_{t,k}) = L_{t-1}(F^L_{t-1,k}) + w(y) = -1$. Since $L_s(F^L_{s,k})$ is monotone nonincreasing function of s, we conclude that $t = t_k$. ∎

The rest of the paper is structured as follows. In section 3, we define a prefix set F and a suffix set H with some special properties, and set $G := A \setminus (F \cup H)$. In sections 4 and 5, we show that for every $1 \le k \le |F|$, $L_t(F^L_{t,k})$ changes (as a function of t) from 0 to -1 at least once, and, for every $1 \le k \le |H|$, $R_t(H^R_{t,k})$ changes from 0 to -1 at least once. Applying Lemmata 2.1 and 2.2, we will obtain that there exist $|F|$ balanced lines through the elements of F and $|H|$ balanced lines through the elements of H. In section 6, we prove that every element of $G = A \setminus (F \cup H)$, gives rise either to a balanced line through an element of G or to a balanced line through an element of $F \cup H$. We show that all of these lines are distinct, so that the number of balanced lines is at least $|F| + |G| + |H| = n$. In section 7, we wrap up the proof of Theorem 1.3, while the last section contains some concluding remarks and generalizations.

3 The definition of F,G, and H

In this section, we continue developing the machinery needed for the proof of Theorem 1.3.

Definition 3.1. *Let $S \subseteq \{1, 2, \ldots, 2n\}$ and $1 \le j \le \lceil \frac{|S|}{2} \rceil$. We say that S has a* barrier *of* order j *if one of the following two conditions is satisfied:*

1. *every element in S has weight $+1$, and*
   (a) *either $L_t(S^L_{t,j}) \ge 0$ and $R_t(S^R_{t,j}) \ge 0$, for every $0 \le t \le \binom{2n}{2}$,*
   (b) *or $L_t(S^L_{t,j}) < 0$ and $R_t(S^R_{t,j}) < 0$, for every $0 \le t \le \binom{2n}{2}$;*
2. *every element in S has weight -1 and*
   (a) *either $L_t(S^L_{t,j}) \le 0$ and $R_t(S^R_{t,j}) \le 0$, for every $0 \le t \le \binom{2n}{2}$,*
   (b) *or $L_t(S^L_{t,j}) > 0$ and $R_t(S^R_{t,j}) > 0$, for every $0 \le t \le \binom{2n}{2}$.*

We say that S has a barrier *if it has a barrier of order j for some index j.*

Consider all (non-empty) sets of the form

$$\{1 \le i \le 2n | u \le i \le v, w(i) = \epsilon\},$$

where $1 \le u < v \le 2n$ and $\epsilon \in \{+1,-1\}$. If at least one of these sets has a barrier, pick one for which $v-u$ is minimum and denote it by A_0. If there is no such set, then let $A_0 = A$.

If A_0 has a barrier, we may assume without loss of generality that condition 1(a) or 2(b) holds in Definition 3.1 (for otherwise we multiply the weight of every element by -1). In other words, there exists $1 \le j_0 \le \lceil \frac{|A_0|}{2} \rceil$ such that

Case 1: every element in A_0 has weight $+1$, and $L_t((A_0)^L_{t,j_0}) \ge 0$ and $R_t((A_0)^R_{t,j_0}) \ge 0$, for every $0 \le t \le \binom{2n}{2}$; or

Case 2: every element in A_0 has weight -1, and $L_t((A_0)^L_{t,j_0}) > 0$ and $R_t((A_0)^R_{t,j_0}) > 0$, for every $0 \le t \le \binom{2n}{2}$.

In either case, we inductively define a decreasing sequence $A_1 \supset A_2 \supset \dots$ of subsets of A as follows.

For every $0 \le t \le \binom{2n}{2}$, let $c_{t,0} := (A_0)^L_{t,j_0}$ and $d_{t,0} := (A_0)^R_{t,j_0}$ (see Definition 1.7). If $A_\mu, c_{0,\mu}, d_{0,\mu}$ have already been defined for all $0 \le \mu < m$, let

$$A_m = \{a \in A | c_{0,m-1} < a < d_{0,m-1}\}.$$

Assume that one of the following conditions is satisfied for some $1 \le j \le \lceil \frac{|A_m|}{2} \rceil$.

Case i: For every $0 \le t \le \binom{2n}{2}$ such that $\max_{0 \le i < m} c_{t,i} \le (A_m)^L_{t,j} \le \min_{0 \le i < m} d_{t,i}$, we have $L_t((A_m)^L_{t,j}) \ge 0$, and for every $0 \le t \le \binom{2n}{2}$ such that $\max_{0 \le i < m} c_{t,i} \le (A_m)^R_{t,j} \le \min_{0 \le i < m} d_{t,i}$, we have $R_t((A_m)^R_{t,j}) \ge 0$.

Case ii: For every $0 \le t \le \binom{2n}{2}$ such that $\max_{0 \le i < m} c_{t,i} \le (A_m)^L_{t,j} \le \min_{0 \le i < m} d_{t,i}$, we have $L_t((A_m)^L_{t,j}) < 0$, and for every $0 \le t \le \binom{2n}{2}$ such that $\max_{0 \le i < m} c_{t,i} \le (A_m)^R_{t,j} \le \min_{0 \le i < m} d_{t,i}$, we have $R_t((A_m)^R_{t,j}) < 0$.

Fix such a number j, set $j_m := j$, and for every $0 \le t \le \binom{2n}{2}$, let $c_{t,m} := (A_m)^L_{t,j_m}$ and $d_{t,m} := (A_m)^R_{t,j_m}$.

If no such j exists or if $A_m = \emptyset$, stop. Let q be the index at which we stopped. That is, the last set we define is A_q. (If A_0 does not have a barrier, then $q = 0$). Note that all elements of $A_1, A_2, \dots, A_q$ have weight $+1$, while the elements of A_0 are all of weight $+1$ or all of weight -1.

If $q > 0$, let

$$\begin{aligned} F &:= \{a \in A | a \le c_{0,q-1}\}, \\ G &:= A_q, \\ H &:= A \setminus (F \cup G) = \{a \in A | a \ge d_{0,q-1}\}. \end{aligned} \tag{1}$$

If $q = 0$, let $F = H = \emptyset$ and $G = A_0 = \{a_1, \dots, a_n\}$.

Clearly, F and H are prefix and suffix sets, respectively.

4 Useful facts about the sets A_m

The following simple observation is crucial for our proposes.

Claim 4.1 (continuity). *Let $S \subseteq \{1, 2, \dots, 2n\}$ and $1 \le i \le |S|$. Then for every $1 \le t \le \binom{2n}{2}$, we have*

1. $|S^L_{t,i} - S^L_{t-1,i}| \le 1$;
2. $|S^R_{t,i} - S^R_{t-1,i}| \le 1$. ∎

Corollary 4.2. *Let $0 \le m < q$. For every $1 \le t \le \binom{2n}{2}$, we have*

1. $|\max_{0 \le i \le m} c_{t,i} - \max_{0 \le i \le m} c_{t-1,i}| \le 1$;
2. $|\min_{0 \le i \le m} d_{t,i} - \min_{0 \le i \le m} d_{t-1,i}| \le 1$. ∎

The aim of this section is to prove the following claim, whose parts 1 and 2 roughly express that in the definition of j_m and A_m at the end of the last section, only Case i can occur. The proof is somewhat tedious but straightforward.

Claim 4.3. *Let $0 \le m < q$ and $0 \le t \le \binom{2n}{2}$.*

1. *If* $\max_{0 \le i < m} c_{t,i} \le c_{t,m} \le \min_{0 \le i < m} d_{t,i}$, *then* $L_t(c_{t,m}) \ge 0$;
2. *if* $\max_{0 \le i < m} c_{t,i} \le d_{t,m} \le \min_{0 \le i < m} d_{t,i}$, *then* $R_t(d_{t,m}) \ge 0$;
3. $\max_{0 \le i \le m} c_{t,i} < \min_{0 \le i \le m} d_{t,i}$.

Proof: We prove the claim by induction on m. Assume $m = 0$. Parts 1 and 2 follow from the fact that A_0 has a barrier and either 1(a) or 2(b) holds in Definition 3.1. Part 3 of the claim, stating that $c_{t,0} < d_{t,0}$, follows from the definitions of those numbers.

Assume that all three parts of the claim have already been verified for all $0 \le i < m$, and we want to prove it for m.

First we prove parts 1 and 2. If either 1 or 2 is not true, then in the definition of A_m Case ii occurs. That is, for every $0 \le t \le \binom{2n}{2}$,

$$\max_{0\le i<m} c_{t,i} \le c_{t,m} \le \min_{0\le i<m} d_{t,i} \implies L_t(c_{t,m}) < 0 \tag{2}$$

$$\max_{0\le i<m} c_{t,i} \le d_{t,m} \le \min_{0\le i<m} d_{t,i} \implies R_t(d_{t,m}) < 0 \tag{3}$$

By definition, $c_{t,m} < d_{t,m}$. Note that it cannot happen that

$$\max_{0\le i<m} c_{t,i} \le c_{t,m} < d_{t,m} \le \min_{0\le i<m} d_{t,i}$$

for every $0 \le t \le \binom{2n}{2}$. Indeed, this would imply that $L_t((A_m)^L_{t,j_m}) = L_t(c_{t,m}) < 0$ and $R_t((A_m)^R_{t,j_m}) = L_t(c_{t,m}) < 0$, for every $0 \le t \le \binom{2n}{2}$. In other words, A_m would have a barrier of order j_m. This would contradict the minimality of $v - u$ in the definition of A_0, because $u \le c_{0,0} < a < d_{0,0} \le v$ holds for every element $a \in A_m$

Therefore, we may assume that there is a minimal t, $0 \le t \le \binom{2n}{2}$, such that $c_{t+1,m} < \max_{0\le i<m} c_{t+1,i}$. (The other case when $d_{t+1,m} > \min_{0\le i<m} d_{t+1,i}$ for some t can be treated similarly.)

By Claim 4.1 and Corollary 4.2, it follows from the minimality of t that one of the following two cases has to occur.

Case a: $c_{t,m} = \max_{0\le i<m} c_{t,i}$;

Case b: $c_{t,m} = \max_{0\le i<m} c_{t,i} + 1$.

Let $0 \le m' < m$ be an index such that $\max_{0\le i<m} c_{t,i} = c_{t,m'}$. Clearly, we have

$$\max_{0\le i<m'} c_{t,i} \quad\le\quad \max_{0\le i<m} c_{t,i} = c_{t,m'}, \tag{4}$$

and, by the induction hypothesis,

$$c_{t,m'} = \max_{0\le i<m} c_{t,i} < \min_{0\le i<m} d_{t,i} \quad\le\quad \min_{0\le i<m'} d_{t,i}. \tag{5}$$

Combining (4) and (5), we obtain

$$\max_{0\le i<m'} c_{t,i} \le c_{t,m'} \le \min_{0\le i<m'} d_{t,i}. \tag{6}$$

By the minimality of t,

$$\max_{0\le i<m} c_{t,i} \le c_{t,m} \le \min_{0\le i<m} d_{t,i}. \tag{7}$$

We discuss Cases a and b separately. In Case a, we have $c_{t,m} = c_{t,m'}$. Using (6) and part 1 of the induction hypothesis for m', we get $L_t(c_{t,m}) = L_t(c_{t,m'}) \ge 0$. In view of (7), this contradicts (2).

In Case b, we have $c_{t,m} = c_{t,m'} + 1$. As before, we get $L_t(c_{t,m'}) \ge 0$. Let $x \in A_{m'}$ be the element of P_t at the position $c_{t,m'} = c_{t,m} - 1$. If all elements of $A_{m'}$ have weight $+1$, then $w(x) = +1$. Therefore,

$$L_t(c_{t,m}) = L_t(c_{t,m'}) + w(x) = L_t(c_{t,m'}) + 1 \ge 1.$$

If $m' = 0$ and all elements of A_0 have weight -1, then, using the fact that A_0 has a barrier, we find that $L_t(c_{t,m'}) = L_t(c_{t,0}) > 0$. Thus,

$$L_t(c_{t,m}) = L_t(c_{t,m'}) + w(x) = L_t(c_{t,m'}) - 1 \ge 0.$$

Hence, in either case $L_t(c_{t,m}) \ge 0$, contradicting (2). This completes the proof of parts 1 and 2.

Next we prove part 3. Assume for a contradiction that there is a minimal t, $0 \le t < \binom{2n}{2}$, such that $\max_{1\le i\le m} c_{t+1,i} \ge \min_{1\le i\le m} d_{t+1,i}$. By the induction hypothesis, $\max_{0\le i<m} c_{t+1,i} < \min_{0\le i<m} d_{t+1,i}$. Therefore, without loss of generality we may assume that $\max_{0\le i<m} c_{t+1,i} < c_{t+1,m}$. (The other case when $d_{t+1,m} < \min_{0\le i<m} d_{t+1,i}$ for some t can be treated similarly).

By the minimality of t and by Corollary 4.2, again there are only two possibilities.

Case a: $\max_{0\le i\le m} c_{t+1,i} = \min_{1\le i\le m} d_{t+1,i}$,

Case b: $\max_{0\le i\le m} c_{t+1,i} = \min_{1\le i\le m} d_{t+1,i} + 1$.

In Case a,

$$\max_{0\le i<m} c_{t+1,i} < c_{t+1,m} \;=\; \min_{0\le i\le m} d_{t+1,i} = \min_{0\le i<m} d_{t+1,i}, \tag{8}$$

where the last equality follows from the fact that $c_{t+1,m} < d_{t+1,m}$.

Let $m' < m$ be such that $\min_{0\le i<m} d_{t+1,i} = d_{t+1,m'}$. Then we have

$$d_{t+1,m'} \;=\; \min_{0\le i<m} d_{t+1,i} \le \min_{0\le i<m'} d_{t+1,i},$$

and, by induction hypothesis,

$$\max_{0\le i<m'} c_{t+1,i} \quad\le\quad \max_{0\le i<m} c_{t+1,i} < \min_{0\le i<m} d_{t+1,i} = d_{t+1,m'}.$$

Combining the last two inequalities, we obtain

$$\max_{0\le i<m'} c_{t+1,i} \le d_{t+1,m'} \le \min_{0\le i<m'} d_{t+1,i}.$$

This, together with part 2 of the claim for m', implies that $R_{t+1}(d_{t+1,m'}) \ge 0$. Let x be the element in P_{t+1} at the position $d_{t+1,m'} = c_{t+1,m}$. By the definition of $c_{t+1,m}$, x belongs to A_m, and therefore $w(x) = +1$. Then

$$L_{t+1}(c_{t+1,m}) = L_{t+1}(d_{t+1,m'}) = -(w(x) + R_{t+1}(d_{t+1,m'})) = -1 - R_{t+1}(d_{t+1,m'}) \le -1$$

where the second equality follows from the fact that the sum of all weights is 0. This, together with (8), contradicts part 1 of the claim.

In Case b, it follows from the minimality of t and Corollary 4.2 that

$$\max_{0\le i\le m} c_{t,i} \quad=\quad \min_{0\le i\le m} d_{t,i} - 1. \tag{9}$$

Since $c_{t+1,m} < d_{t+1,m}$, we have

$$c_{t+1,m} \quad=\quad \max_{0\le i\le m} c_{t+1,i} = \min_{1\le i\le m} d_{t+1,i} + 1 = \min_{1\le i<m} d_{t+1,i} + 1,$$

and, by the induction hypothesis,

$$\max_{0\le i<m} c_{t+1,i} + 1 \quad<\quad \min_{0\le i<m} d_{t+1,i} + 1 = c_{t+1,m}$$

Therefore, $\max_{0\le i<m} c_{t+1,i} + 2 \le c_{t+1,m}$ and, by Claim 4.1, we obtain $\max_{0\le i<m} c_{t,i} \le c_{t,m}$. This, together with (9), implies that

$$c_{t,m} \quad=\quad \max_{0\le i\le m} c_{t,i} = \min_{0\le i\le m} d_{t,i} - 1 = d_{t,m'} - 1, \tag{10}$$

where $m' \le m$ is such that $\min_{0\le i\le m} d_{t,i} = d_{t,m'}$. Then we have

$$\max_{0\le i<m'} c_{t,i} \quad\le\quad \max_{0\le i\le m} c_{t,i} < d_{t,m'} = \min_{0\le i\le m} d_{t,i} \le \min_{0\le i<m'} d_{t,i}.$$

Here the second inequality follows from (10). So, by part 1 of the claim for m',

$$R_t(d_{t,m'}) \geq 0.$$

Let $x \in A_{m'}$ be the element in P_t at the position $d_{t,m'}$. In view of (10),

$$R_t(c_{t,m}) = R_t(d_{t,m'}) + w(x).$$

If all elements of $A_{m'}$ have weight $+1$, then $w(x) = +1$, and thus

$$R_t(c_{t,m}) = R_t(d_{t,m'}) + 1 \geq 1.$$

If $m' = 0$ and all elements of A_0 have weight -1, then

$$R_t(c_{t,m}) = R_t(d_{t,0}) - 1 \geq 0,$$

because $R_t(d_{t,0}) = R_t((A_0)^R_{t,j_0}) > 0$, by the definition of A_0. In either case, $R_t(c_{t,m}) \geq 0$. Let $y \in A_m$ be the element in P_t at the position $c_{t,m}$. Then $w(y) = +1$, therefore

$$L_t(c_{t,m}) = -(w(y) + R_t(c_{t,m})) = -(1 + R_t(c_{t,m})) < 0.$$

This, combined with (10), contradicts part 1 of the claim, completing the proof. ∎

Notation 4.4. *For every $0 \leq t \leq \binom{2n}{2}$, let $C_t = \max_{0 \leq i < q} c_{t,i}$ and $D_t = \min_{0 \leq i < q} d_{t,i}$.*

Corollary 4.5. *For every $0 \leq t \leq \binom{2n}{2}$, we have*

1. *$L_t(C_t) \geq 0$ and $R_t(D_t) \geq 0$,*
2. *$L_t(C_t + 1) \geq 0$ and $R_t(D_t - 1) \geq 0$.*

Proof: Fix $0 \leq t \leq \binom{2n}{2}$. We prove only the first assertion of part 1; the proof of the second assertion is very similar. Choose $0 \leq m < q$ so that $C_t = c_{t,m}$. Then we have

$$\max_{0 \leq i < m} c_{t,i} \leq \max_{0 \leq i < q} c_{t,i} = c_{t,m} = \max_{0 \leq i < q} c_{t,i} < \min_{0 \leq i < q} d_{t,i} \leq \min_{0 \leq i < m} d_{t,i},$$

where the second inequality follows from part 3 of Claim 4.3. Thus, part 1 of Claim 4.3 immediately implies that

$$L_t(C_t) = L_t(c_{t,m}) \geq 0.$$

Next we prove the first assertion of part 2. Again, choose $0 \leq m < q$ so that $C_t = c_{t,m}$. By part 1, $L_t(c_{t,m}) \geq 0$. Let $x \in A_m$ be the element in P_t at the position $c_{t,m}$. If $m \neq 0$ or $m = 0$ and all elements of A_0 have weight $+1$, then $w(x) = +1$. Therefore,

$$L_t(C_t + 1) = L_t(c_{t,m} + 1) = L_t(c_{t,m}) + w(x) = L_t(c_{t,m}) + 1 \geq 1.$$

If $m = 0$ and all elements of A_0 have weight -1, then $w(x) = -1$. Recall that, according to the definition of A_0 and $c_{t,0}$, we have $L_t(c_{t,0}) > 0$. Thus,

$$L_t(C_t + 1) = L_t(c_{t,0} + 1) = L_t(c_{t,0}) + w(x) = L_t(c_{t,0}) - 1 \geq 0,$$

as required. The second assertion of part 2 can be verified analogously. ∎

5 Balanced lines through the points of F and H

Using Notation 4.4, we can rewrite the definition of F,G, and H (at the end of section 3) as follows.

$$\begin{aligned} F &= \{i \in A | i \leq C_0\}, \\ G &= A_q = A \setminus (F \cup H), \\ H &= \{i \in A | i \geq D_0\}. \end{aligned} \tag{11}$$

In this section we show that for every $1 \leq k \leq |F|$, as t goes from 0 to $\binom{2n}{2}$, $L_t(F^L_{t,k})$ changes from 0 to -1 at least once. Similarly, for every $1 \leq k \leq |H|$, $R_t(H^R_{t,k})$ changes from 0 to -1 at least once. Thus, Lemmata 2.1 and 2.2 imply that the number of balanced lines passing through some element of F (and H) is at least $|F|$ (at least $|H|$, respectively).

Definition 5.1. *For any $1 \leq k \leq |F|$, let $t(F,k)$ denote the minimal t such that $F^L_{t,k} \geq C_t$, and let $T(F,k)$ denote the maximal t such that $F^L_{t,k} \leq D_t$.*

Similarly, for any $1 \leq k \leq |H|$, let $t(H,k)$ (and $T(H,k)$) denote the minimal t such that $H^R_{t,k} \leq D_t$ (the maximal t such that $H^R_{t,k} \geq C_t$, respectively).

First we show that the above definition is correct.

Claim 5.2. *The numbers $t(F,k), T(F,k), t(H,k), T(H,k)$ exist.*

Proof: We prove only the existence of $t(F,k)$ and $T(F,k)$. By (11), we have $F^L_{0,k} \leq C_0$, for every $1 \leq k \leq |F|$. It follows from part 3 of Claim 4.3, that $C_t < D_t$, for every $0 \leq t \leq \binom{2n}{2}$. Therefore, it suffices to show that $F^L_{\binom{2n}{2},k} \geq D_{\binom{2n}{2}}$.

Assume $0 \leq m < q$, where q is the same as in (1). Denote by x the element at the position $c_{0,m} = (A_m)^L_{0,j_m}$ in P_0. Then x is the j_m'th leftmost element of A_m in P_0. $P_{\binom{2n}{2}}$ is a reversed copy of P_0, i.e., $P_{\binom{2n}{2}} = (2n, 2n-1, \dots, 2, 1)$. Therefore, in $P_{\binom{2n}{2}}$, x is the j_m'th rightmost element of A_m. In other words, x is at position $d_{\binom{2n}{2},m} = (A_m)^R_{\binom{2n}{2},j_m}$ in $P_{\binom{2n}{2}}$.

For every $0 \leq m < q$, let x_m denote the element at position $c_{0,m}$ in P_0. By the definition of the sets $A_0, A_1, \dots, A_{q-1}$, we have $x_0 < x_1 < \dots < x_{q-1}$. Thus, for every $0 \leq m < q$, x_m is at position $d_{\binom{2n}{2},m}$ in $P_{\binom{2n}{2}}$. Since in $P_{\binom{2n}{2}}$ the numbers $x_0, \dots, x_{q-1}$ are in reversed order, we may conclude that $d_{\binom{2n}{2},q-1} < d_{\binom{2n}{2},q-2} < \dots < d_{\binom{2n}{2},0}$.

Let $y \in F$. By the definition of F, we have $y \leq C_0 = c_{0,q-1}$. Therefore, $y \leq x_{q-1}$ and hence y is at a position greater or equal to the position of x_{q-1} in $P_{\binom{2n}{2}}$, which is $d_{\binom{2n}{2},q-1} = D_{\binom{2n}{2}}$. In particular, it follows that $F^L_{\binom{2n}{2},k} \geq D_{\binom{2n}{2}}$ for every $1 \leq k \leq |F|$. ∎

Definition 5.3. *For any $1 \le k \le |F|$, let $\tau(F,k)$ denote the number of different values of t for which $t(F,k) < t \le T(F,k)$, and which satisfy $L_{t-1}(F^L_{t-1,k}) = -1$ and $L_t(F^L_{t,k}) = 0$.*

Similarly, for any $1 \le k \le |H|$, let $\tau(H,k)$ denote the number of different values of t for which $t(F,k) < t \le T(F,k)$, and which satisfy $R_{t-1}(H^R_{t-1,k}) = -1$ and $R_t(H^R_{t,k}) = 0$.

Lemma 5.4. *For any $1 \le k \le |F|$, there are at least $1+\tau(F,k)$ balanced lines l meeting the following two requirements.*

1. *l passes through a point of F,*
2. *there are exactly $k-1$ points of F in the open half-plane which is to the left of l.*

Proof: According to Lemma 2.1 (and using the continuity of $L_t(F^L_{t,k})$, as a function of t), it is enough to show that $L_{t(F,k)}(F^L_{t(F,k),k}) \ge 0$ and $L_{T(F,k)}(F^L_{T(F,k),k}) < 0$.

Let $t_0 = t(F,k)$. By the definition of $t(F,k)$ we have, $F^L_{t_0,k} \ge C_{t_0}$. If $t_0 = 0$, then $F^L_{t_0,k} = C_{t_0}$ (for $F^L_{0,k} \le C_0$). If $t_0 > 0$ then, by the minimality of $t(F,k)$, $F^L_{t_0-1,k} < C_{t_0-1}$. Therefore, by Corollary 4.2, either $F^L_{t_0,k} = C_{t_0}$ or $F^L_{t_0,k} = C_{t_0}+1$.

We conclude that in both cases either $F^L_{t_0,k} = C_{t_0}$ or $F^L_{t_0,k} = C_{t_0}+1$. In either case, we use Corollary 4.5, to argue that $L_{t_0}(F^L_{t_0,k}) \ge 0$.

Similarly, let $t_1 = T(F,k)$. Then, by the maximality of $T(F,k)$, either $F^L_{t_1,k} = D_{t_1}$ or $F^L_{t_1,k} = D_{t_1}-1$. In either case, Corollary 4.5 implies $R_{t_1}(F^L_{t_1,k}) \ge 0$. Let x be the element in P_{t_1} at the position $F^L_{t_1,k}$. Then $x \in F$ and hence $w(x) = 1$. Therefore,

$$L_{t_1}(F^L_{t_1,k}) = -(w(x) + R_{t_1}(F^L_{t_1,k})) = -1 - R_{t_1}(F^L_{t_1,k}) < 0. \quad \blacksquare$$

Similarly, we have

Lemma 5.5. *For any $1 \le k \le |H|$, there are at least $1+\tau(H,k)$ balanced lines l meeting the following two requirements.*

1. *l passes through a point of H,*
2. *there are exactly $k-1$ points of H in the open half-plane which is to the right of l.*

6 The contribution of G

In this section, we estimate from below the contribution of G to the number of balanced lines. We prove (Lemma 6.2) that there are at least $|G|$ different values of t, for which either $L_t(G^L_{t,k})$ or $R_t(G^R_{t,k})$ changes from -1 to 0 or vice versa (for some k, as we go from $t-1$ to t). Then we show (Claim 6.4) that for each such t, either l_t is a balanced line through an element of G or $\sum_{1\le k\le |F|}\tau(F,k) + \sum_{1\le k\le |H|}\tau(H,k)$ increased by 1. However, in the latter case we find a new balanced line through an element of $F \cup H$.

We need an auxiliary lemma.

Lemma 6.1. *Let* $1 \leq k \leq \lceil \frac{|G|}{2} \rceil$ *and* $t_0 < t_1$*. Suppose that* $C_{t_0} \leq G^L_{t_0,k} \leq D_{t_0}$ *and* $C_{t_1} \leq G^L_{t_1,k} \leq D_{t_1}$*.*

(*) $t_0 < t \leq t_1$*,* $C_{t-1} \leq G^L_{t-1,k} \leq D_{t-1}$*, and* $C_t \leq G^L_{t,k} \leq D_t$*.*

(a) If $L_{t_0}(G^L_{t_0,k}) \geq 0$ *and* $L_{t_1}(G^L_{t_1,k}) < 0$*, then there is an integer* t *satisfying*

$$t_0 < t \leq t_1, \quad C_{t-1} \leq G^L_{t-1,k} \leq D_{t-1}, \quad and C_t \leq G^L_{t,k} \leq D_t \tag{12}$$

such that $L_{t-1}(G^L_{t-1,k}) = 0$ *and* $L_t(G^L_{t,k}) = -1$*;*

(b) if $L_{t_0}(G^L_{t_0,k}) < 0$ *and* $L_{t_1}(G^L_{t_1,k}) \geq 0$*, then there is an integer* t *satisfying 12 such that* $L_{t-1}(G^L_{t-1,k}) = -1$ *and* $L_t(G^L_{t,k}) = 0$*;*

(c) if $R_{t_0}(G^R_{t_0,k}) \geq 0$ *and* $R_{t_1}(G^R_{t_1,k}) < 0$*, then there is an integer* t *satisfying 12 such that* $R_{t-1}(G^R_{t-1,k}) = 0$ *and* $R_t(G^R_{t,k}) = -1$*;*

(d) if $R_{t_0}(G^R_{t_0,k}) < 0$ *and* $R_{t_1}(G^R_{t_1,k}) \geq 0$*, then there is an integer* t *satisfying 12 such that* $R_{t-1}(G^R_{t-1,k}) = -1$ *and* $R_t(G^R_{t,k}) = 0$*.*

Proof: By symmetry, it is enough to discuss the case $L_{t_0}(G^L_{t_0,k}) \geq 0$ and $L_{t_1}(G^L_{t_1,k}) < 0$. (The other cases can be treated similarly.)

Let t be the minimum integer in $(t_0, t_1]$, for which $L_t(G^L_{t,i}) < 0$ and $C_t \leq G^L_{t,k} \leq D_t$. We show that t meets the requirements of the lemma.

If $C_{t-1} \leq G^L_{t-1,k} \leq D_{t-1}$, then $L_{t-1}(G^L_{t-1,k}) = 0$, by the minimality of t, and we are done.

Otherwise, we distinguish two cases.

Case 1: $G^L_{t-1,k} < C_{t-1}$;

Case 2: $G^L_{t-1,k} > D_{t-1}$.

Since $C_t \leq G^L_{t,k} \leq D_t$, it follows from Corollary 4.2 that in Case 1 either $G^L_{t,k} = C_t$ or $G^L_{t,k} = C_t + 1$; and in Case 2 either $G^L_{t,k} = D_t$ or $G^L_{t,k} = D_t - 1$.

Case 1 is impossible, because $L_t(G^L_{t,k}) < 0$, while, by Corollary 4.5, $L_t(C_t) \geq 0$ and $L_t(C_t + 1) \geq 0$. Contradiction.

In Case 2, let t' be the maximum integer in $[t_0, t-1)$ such that $G^L_{t',k} \leq D_{t'}$. By the maximality of t' and by Corollary 4.2, $G^L_{t',k}$ is either $D_{t'}$ or $D_{t'} - 1$. In either case, Corollary 4.5 implies that $R_{t'}(G^L_{t',i}) \geq 0$. Therefore, denoting by x the element in $P_{t'}$ at position $G^L_{t',k}$, we have

$$L_{t'}(G^L_{t',k}) = -(w(x) + R_{t'}(G^L_{t',k})) = -(1 + R_{t'}(G^L_{t',k})) < 0.$$

Moreover, we have $C_{t'} \leq G^L_{t',k} \leq D_{t'}$. Thus, t' contradicts the minimality of t. (Observe that $t' \neq t_0$, because $L_{t'}(G^L_{t',k}) < 0$, while $L_{t_0}(G^L_{t_0,k}) \geq 0$.) ∎

Lemma 6.2. *Let $1 \leq k \leq \lfloor \frac{|G|}{2} \rfloor$. Then there exist $0 < t^1_k, t^2_k \leq \binom{2n}{2}$, $t^1_k \neq t^2_k$, such that for $t \in \{t^1_k, t^2_k\}$, precisely one of the following two conditions is satisfied.*

1. *$\{L_{t-1}(G^L_{t-1,k}), L_t(G^L_{t,k})\} = \{0, -1\}$, $C_{t-1} \leq G^L_{t-1,k} \leq D_{t-1}$, and $C_t \leq G^L_{t,k} \leq D_t$;*
2. *$\{R_{t-1}(G^R_{t-1,k}), R_t(G^R_{t,k})\} = \{0, -1\}$, $C_{t-1} \leq G^R_{t-1,k} \leq D_{t-1}$, and $C_t \leq G^R_{t,k} \leq D_t$.*

Furthermore, if $|G|$ is odd and $k = \frac{|G|+1}{2}$, then there exists at least one $t = t_k$, $0 \leq t \leq \binom{2n}{2}$, satisfying condition 1 or 2.

All numbers t^1_k, t^2_k, t_k having the above properties are different for different values of k.

Proof: Suppose first that $L_0(G^L_{0,k}) \geq 0$ and $R_0(G^R_{0,k}) < 0$. Since $P_{\binom{2n}{2}}$ is a reversed copy of P_0, we have that $L_{\binom{2n}{2}}(G^L_{\binom{2n}{2},k}) = R_0(G^R_{0,k}) < 0$. By the definition of G, for every $1 \leq j \leq |G|$, $C_0 \leq G^L_{0,j} \leq D_0$ so that $C_{\binom{2n}{2}} \leq G^L_{\binom{2n}{2},j} \leq D_{\binom{2n}{2}}$. Therefore, Lemma 6.1 implies that there exists t^1_k for which condition 1 of Lemma 6.2 holds.

To prove the existence of t^2_k, note that $R_{\binom{2n}{2}}(G^R_{\binom{2n}{2},k}) = L_0(G^L_{0,k}) \geq 0$. Now Lemma 6.1 implies that there exists t^2_k satisfying condition 2 of Lemma 6.2.

Next, suppose that $L_0(G^L_{0,k}) \geq 0$ and $R_0(G^R_{0,k}) \geq 0$.

Then $L_{\binom{2n}{2}}(G^L_{\binom{2n}{2},k}) = R_0(G^R_{0,k}) \geq 0$ and $R_{\binom{2n}{2}}(G^R_{\binom{2n}{2},k}) = L_0(G^L_{0,k}) \geq 0$. By the construction of G, at least one of the following two conditions is satisfied:

(i) there exist t_0, t_1 such that $L_{t_0}(G^L_{t_0,k}) \geq 0$, $L_{t_1}(G^L_{t_1,k}) < 0$, $C_{t_0} \leq G^L_{t_0,k} \leq D_{t_0}$, and $C_{t_1} \leq G^L_{t_1,k} \leq D_{t_1}$;

(ii) there exist t_0, t_1 such that $R_{t_0}(G^R_{t_0,k}) \geq 0$, $R_{t_1}(G^R_{t_1,k}) < 0$, $C_{t_0} \leq G^R_{t_0,k} \leq D_{t_0}$, and $C_{t_1} \leq G^R_{t_1,k} \leq D_{t_1}$.

If (i) holds, then part (a) and (b) of Lemma 6.1 imply that there exist t_k^1 and t_k^2, $0 \leq t_k^1 < t_1 < t_k^2 \leq \binom{2n}{2}$, for which condition 1 of Lemma 6.2 is satisfied.

If (ii) holds then, similarly, condition 2 of Lemma 6.2 can be derived from parts (c) and (d) of Lemma 6.1.

The remaining cases can be settled in the same way. Note that the above argument also applies when $k = \frac{|G|+1}{2}$, but in this case t_k^1 and t_k^2 may coincide.

We prove the last statement of Lemma 6.2 by contradiction. Suppose, e.g., that there are two integers $1 \leq k \neq k' \leq \lceil \frac{|G|}{2} \rceil$ such that $t_k \in \{t_k^1, t_k^2\}$, $t_{k'} \in \{t_{k'}^1, t_{k'}^2\}$, and $t_k = t_{k'} = t$. If t satisfies condition 1 of the lemma, then $L_{t-1}(G_{t-1,k}^L) \neq L_t(G_{t,k}^L)$. In this case, l_t passes through a unique element of G. Indeed, if l_t passed through two elements of G or no element of G, we would have $G_{t-1,k}^L = G_{t,k}^L$ and hence also $L_{t-1}(G_{t-1,k}^L) = L_t(G_{t,k}^L)$. Moreover, this unique element of G is at position $G_{t-1,k}^L$ in P_{t-1}.

Similarly, if condition 2 is satisfied, then l_t passes through a unique element of G, which is at position $G_{t-1,k}^R$ in P_{t-1}. Therefore, if $t = t_k = t_{k'}$, we have $\{G_{t-1,k}^L, G_{t-1,k}^R\} \cap \{G_{t-1,k'}^L, G_{t-1,k'}^R\} \neq \emptyset$, which is a contradiction, as $1 \leq k \neq k' \leq \lceil \frac{|G|}{2} \rceil$. ∎

Notation 6.3. *For any $S \subseteq \{1, 2, \ldots, 2n\}$, let* bal$(S)$ *denote the number of balanced lines passing through at least one point of S.*

Claim 6.4. $|G| \leq \sum_{1 \leq k \leq |F|} \tau(F, k) + \sum_{1 \leq k \leq |H|} \tau(H, k) + \text{bal}(G)$

Proof: Let $1 \leq k \leq \lceil \frac{|G|}{2} \rceil$, and let t be one of the values t_k^1, t_k^2, whose existence is guaranteed by Lemma 6.2. (Note that in case $k = \frac{|G|+1}{2}$ there is only one such value.)

Then $C_{t-1} \leq G_{t-1,k}^L \leq D_{t-1}$, and $C_t \leq G_{t,k}^L \leq D_t$. There are four possibilities:

1. (a) $L_{t-1}(G_{t-1,k}^L) = 0$ and $L_t(G_{t,k}^L) = -1$,
   (b) $L_{t-1}(G_{t-1,k}^L) = -1$ and $L_t(G_{t,k}^L) = 0$,
2. (a) $R_{t-1}(G_{t-1,k}^R) = 0$ and $R_t(G_{t,k}^R) = -1$,
   (b) $R_{t-1}(G_{t-1,k}^R) = -1$ and $R_t(G_{t,k}^R) = 0$.

For simplicity, we consider only case 1(a). Let x denote the element at position $G_{t-1,k}^L$ in P_{t-1}. Since $x \in G$, we have $w(x) = +1$. P_{t-1} and P_t differ in two consecutive places; one of them is occupied by x. Let y denote the element at the other place. Obviously, l_t passes through x and y. We distinguish two cases.

Case 1: $w(y) = -1$.

Clearly, $y \notin G$, so x is at position $G_{t,k}^L$ in P_t. Since $L_t(G_{t,k}^L) < L_{t-1}(G_{t-1,k}^L)$, it follows that $y > x$. That is, $L_t(G_{t,k}^L) = L_{t-1}(G_{t-1,k}^L) + w(y)$. Consequently, the sum of the weights of the points of V in the open half-plane to the left of l_t, is 0. Since $w(x) + w(y) = 0$, l_t must be a balanced line.

Case 2: $w(y) = +1$.

Now $y \notin G$, for otherwise $L_t(G^L_{t,k}) = L_{t-1}(G^L_{t-1,k})$.

Using the fact that $L_t(G^L_{t,k}) < L_{t-1}(G^L_{t-1,k})$, we obtain that $y < x$. That is $L_t(G^L_{t,k}) = L_{t-1}(G^L_{t-1,k}) - w(y)$. Since $y \notin G$ and $y < x$, we have $y \in F$. Let $1 \leq s \leq |F|$ denote the integer for which y is the s-th leftmost element of F in P_{t-1} and hence also in P_t. Now it follows that $L_{t-1}(F^L_{t-1,s}) = -1$ and $L_t(F^L_{t,s}) = 0$. We show that $t(F, s) < t \leq T(F, s)$, which implies that when x and y are swapped, $\tau(F, s)$ increases by 1.

To see that $t(F, s) < t$, it is enough to prove that $C_{t-1} \leq F^L_{t-1,s}$. Since $L_t(G^L_{t,k}) = -1$, using Corrolary 4.5 and the fact that $C_t \leq G^L_{t,k}$ we have $C_t+2 \leq G^L_{t,k}$. Now $G^L_{t,k} = F^L_{t,s}-1$, so that $C_t + 3 \leq F^L_{t,s}$. It follows from Claim 4.1 and Corrolary 4.2 that $C_{t-1} < F^L_{t-1,s}$.

To see that $t \leq T(F, s)$, it is enough to prove that $F^L_{t,s} \leq D_t$. Now $R_{t-1}(G^L_{t-1,k}) = -(L_{t-1}(G^L_{t-1,k}) + w(x)) < 0$. Since $G^L_{t-1,k} \leq D_{t-1}$, it follows from Corrolary 4.5 that $G^L_{t-1,k} \leq D_{t-1} - 2$. We have $F^L_{t-1,s} = G^L_{t-1,k} - 1$, so that $F^L_{t-1,s} \leq D_{t-1} - 3$. It follows from Claim 4.1 and Corrolary 4.2 that $F^L_{t,s} \leq D_t - 1$.

Summarizing, we have shown that for every value of t, whose existence is guaranteed by Lemma 6.2, either l_t is a distinct balanced line through an element of G, or t contributes 1 to the sum $\sum_{1\leq k\leq|F|} \tau(F, k) + \sum_{1\leq k\leq|H|} \tau(H, k)$. ∎

7 Proof of the Theorem 1.3

Now we are in a position to complete the proof of Theorem 1.3. Since $F \cup G \cup H$ is the set of all elements of weight +1, by Claim 1.2 we have that the number of balanced lines is equal to $\mathrm{bal}(F) + \mathrm{bal}(H) + \mathrm{bal}(G)$. By Lemmata 5.4 and 5.5, we have

$$\mathrm{bal}(F) \geq \sum_{1\leq k\leq|F|} (1 + \tau(F, k)), \quad \mathrm{bal}(H) \geq \sum_{1\leq k\leq|H|} (1 + \tau(H, k)).$$

Therefore, in view of Claim 6.4, the number of balanced lines is

$$\begin{aligned}
\mathrm{bal}(F) + \mathrm{bal}(H) + \mathrm{bal}(G) &\geq \sum_{1\leq k\leq|F|} (1 + \tau(F, k)) + \sum_{1\leq k\leq|H|} (1 + \tau(H, k)) + \mathrm{bal}(G) \\
&= |F| + |H| + \left(\sum_{1\leq k\leq|F|} \tau(F, k) + \sum_{1\leq k\leq|H|} \tau(H, k) + \mathrm{bal}(G) \right) \\
&\geq |F| + |H| + |G| = n. \blacksquare
\end{aligned}$$

8 Concluding remarks

Theorem 1.3 does not remain true without assuming that the points are in general position. It is not hard to construct sets of n points of weight $+1$ and n points of weight -1 which determine *no* balanced line.

Theorem 1.3 can be rephrased in the following *dual* form. Consider n lines of weight $+1$ and n lines of weight -1 in general position in the plane, i.e., no three of them pass through the same point, no two are parallel, and none of them is vertical (parallel to the y-axis). Then they determine at least n intersection points p with the property that the sum of the weights of all lines above p, as well as the sum of the weights of all lines below p, is equal to zero. This statement can also be formulated for x-monotone *pseudo-lines* instead of lines (a pseudo-line is called x-monotone if every vertical line intersects it in precisely one point). This version remains valid, because as we sweep the plane by a vertical line from left to right, the order in which it meets the pseudo-lines determines a flip array, and our proof applies.

Let V be a set of points in general position in the plane, having an even number of elements. A line l connecting two points of V is called a *halving line*, if it cuts V into two equal halves, i.e., if both open half-planes bounded by l contain precisely $|V|/2 - 1$ elements of V.

The following simple fact is an easy consequence of the Ham-sandwich Theorem (for a similar argument, see [AA89]).

Claim 8.1. *Let V consist of n points of weight $+1$ and n points of weight -1 in general position in the plane. If n is odd, then V permits a balanced halving line l.*

Proof: Replace each point $v \in V$ by a disc of area $1/N$ centered at v, where N is a sufficiently large integer. Let D^+ and D^- denote the union of all discs which correspond to the elements of V with positive and negative weights, respectively. By the Ham-sandwich Theorem, there is a straight line $l(N)$ such that the area of the intersection of D^+ with any half-plane bounded by $l(N)$ is $n/(2N)$, and the same is true for D^-. Choose an infinite sequence $N(1) < N(2) < \ldots$ such that the corresponding lines $l(N_i)$ converge to a straight line l, as i tends to infinity. Clearly, l must connect a point of positive weight weight with a point of negative weight, and it meets the requirements in the claim. ∎

It is not hard to come up with a point set V satisfying the conditions in Lemma 8.1, which permits only one balanced halving line. (See Figure 2.)

The above argument easily generalizes to any d-dimensional set V in general position, whose elements are colored with d colors. However, the analogue of Theorem 1.3 does not hold in 3 and higher dimensions.

Definition 8.2. *A set of points in d-space is said to be in* general position*, if no $d+1$ of them lie on a hyperplane.*

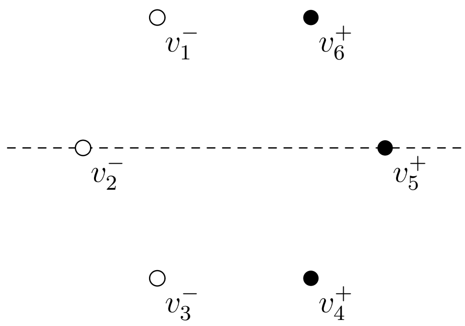

Figure 2: A 2-colored point set with a unique balanced halving line

Let $U = U_1 \cup \ldots \cup U_d$ be a set of dn points in general position in d-space, where each U_i consists of n points and is called a color class.

A hyperplane h determined by (d elements of) V is called balanced *if each open half-space bounded by h contains the same number of elements from each color class.*

Obviously, all points a balanced hyperplane are of different colors. By straightforward generalization of the proof of Claim 8.1, we also obtain that if n is odd, then $U = U_1 \cup \ldots \cup U_d$ always permits at least one balanced halving hyperplane.

Claim 8.3. *For every $d \geq 3$, there exists a set U of dn points in general position in d-space, which consists of d color classes of size n and satisfies the following condition:*

(i) if n is even, then U does not permit a balanced hyperplane;

(ii) if n is odd, then U permits precisely one balanced hyperplane.

Proof: We present the construction only for $d = 3$; the other constructions are very similar.

Suppose first that n is even. Let $\{a, b, c, d\}$ be the vertex set of a regular tetrahedron centered at o. Replace a, b, c, d and o by five point sets, A, B, C, D, and O, respectively. Suppose that each of these sets is equally spaced along a line parallel to od, with a sufficiently small distance $\varepsilon > 0$, and let $|A| = |B| = |C| = |D| = n/2$, and $|O| = n$. Finally, slightly perturb the points so that $A \cup B \cup C \cup D \cup O$ will be in general position.

Let $U_1 := A \cup B$, $U_2 := C \cup D$, and $U_3 := O$. Suppose, in order to obtain a contradiction, that $U := U_1 \cup U_2 \cup U_3$ permits a balanced hyperplane h. Clearly, h must pass through three points of different colors, say, $u \in A$, $v \in C$, and $w \in O$. Now B and D are on different sides of h, which implies that both open half-spaces bounded by h must contain at least $n/2$ points of each color. Counting the points u, v, and w belonging to h, each color class has at least $n + 1$ elements, a contradiction.

If n is odd, then the construction is the same, except that $|A| = |C| = (n+1)/2$ and $|B| = |D| = (n-1)/2$. Now a balanced hyperplane h must pass through one element in each of the sets A, C, and O, say, u, v, and w, resp. Moreover, since there are at least $(n-1)/2$ elements of U_2 in the open half-space opposite to D, v must be the last point of C in the direction od. Similarly, u is the last point of A in the same direction, and w is also uniquely determined. ∎

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