Crossroads in Flatland

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Drawing is one of the most ancient human activities. Our ancestors drew their pictures (pictographs or, simply, "*graphs*") on walls of caves, nowadays we use mostly computer screens for this purpose. From the mathematical point of view, there is not much difference: both surfaces are "flat," they are topologically equivalent.

1 Crossings – the Brick Factory Problem

Every graph consists of *vertices* and *edges*. The vertex set of a graph G is a finite set $V(G)$, and its edge set, $E(G)$, is a collection of unordered pairs from $V(G)$. By a *drawing* of G, we mean a representation of G in the plane such that each vertex is represented by a distinct point and each edge by a simple (non-selfintersecting) continuous arc connecting the corresponding two points. If it is clear whether we talk about an "abstract" graph G or its planar representation, these points and arcs will also be called vertices and edges, respectively. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its endpoints, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point.

Every graph has many different drawings. If G can be drawn in such a way that no two edges cross each other, then G is *planar*. According to an observation of István Fáry [11], if G is planar then it has a drawing, in which every edge is represented by a straight-line segment.

Not every graph is planar. It is well known that K_5, the *complete graph* with 5 vertices, and $K_{3,3}$, the *complete bipartite graph* with 3 vertices in its classes are not planar. According to Kuratowski's famous theorem, a graph is planar if and only if it has no subgraph which can be obtained from

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K_5 or from $K_{3,3}$ by subdividing some (or all) of itsedges with distinct new vertices. In the next section, we give a completely different representation of planar graphs (see Theorem 2.3).

If G is not planar then it cannot be drawn in the plane without crossing. Paul Turán [38] raised the following problem: find a drawing of G, for which the number of crossings is minimum. This number is called the *crossing number* of G and is denoted by $\mathrm{CR}(G)$. More precisely, Turán's (still unsolved) original problem was to determine $\mathrm{CR}(K_{n,m})$, for every $n, m \geq 3$. According to an assertion of Zarankiewicz, which was down-graded from theorem to conjecture [14], we have

$$\mathrm{CR}(K_{n,m}) = \lfloor \frac{m}{2} \rfloor \cdot \lfloor \frac{m-1}{2} \rfloor \cdot \lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{n-1}{2} \rfloor,$$

but we do not even know the limits

$$\lim_{n \to \infty} \frac{\mathrm{CR}(K_{n,n})}{n^4}, \quad \lim_{n \to \infty} \frac{\mathrm{CR}(K_n)}{n^4}$$

(cf. [34], [20]).

Turán used to refer to this question as the "brick factory problem," because it occurred to him at a factory yard, where, as forced labour during World War II, he moved waggons filled with bricks from kilns to storage places. According to his recollections, it was not a very tough job, except that they had to push much harder at the crossings. Had this been the only "practical application" of crossing numbers, much fewer people would have tried to estimate $\mathrm{CR}(G)$ during the past quarter of a century. In the early eighties, it turned out that the chip area required for the realization (VLSI layout) of an electrical circuit is closely related to the crossing number of the underlying graph [22]. This discovery gave an impetus to research in the subject.

2 Thrackles – Conway's Conjecture

A drawing of a graph is called a *thrackle*, if any two edges which do not share an endpoint cross precisely once, and if two edges share an endpoint then they have no other point in common.

It is easy to verify that e.g. C_4, a cycle of length 4, cannot be drawn as a thrackle, but any other cycle can [41]. If a graph cannot be drawn as a thrackle, then the same is true for all graphs that contain it as a subgraph. Thus, a thrackle does not contain a cycle of length 4, and, according to an old theorem of Erdős in extremal graph theory, the number of its edges cannot exceed $n^{3/2}$, where n denotes the number of its vertices.

The following old conjecture states much more.

Conjecture (J. Conway). *Every thrackle has at most as many edges as vertices.*

The first upper bound on the number of edges of a thrackle, which is linear in n, was found in [23].

2.1 Theorem ([23]). *Every thrackle has at most twice as many edges as vertices.*

Thrackle and planar graph are, in a certain sense, opposite notions: in the former any two edges intersect, in the latter there is no crossing pair of edges. Yet the next theorem shows how similar these concepts are.

A drawing of a graph is said to be a *generalized thrackle* if every pair of its edges intersect an odd number of times. Here the common endpoint of two edges also counts as a point of intersection. Clearly, every thrackle is a generalized thrackle, but not the other way around. For example, a cycle of length 4 can be drawn as a generalized thrackle, but not as a thrackle.

2.2 Theorem ([23]). *A bipartite graph can be drawn as a thrackle if and only if it is planar.*

According to an old observation of Erdős, every graph has a bipartite subgraph which contains at least half of its edges. Clearly, every planar graph of $n \geq 3$ vertices has at most $2n - 4$ edges. Hence, Theorem 2.2 immediately implies that every thrackle with $n \geq 3$ vertices has at most $2(2n - 4) = 4n - 8$ edges. This statement is slightly weaker than Theorem 2.1.

In a drawing of a graph, a triple of internally disjoint paths $(P_1(u, v)$, $P_2(u, v)$, $P_3(u, v))$ between the same pair of vertices (u, v) is called a *trifurcation.* (The three paths cannot have any vertices in common, other than u and v, but they can cross at points different from their vertices.) A trifurcation $(P_1(u, v)$, $P_2(u, v)$, $P_3(u, v))$ is said to be a *converter* if the cyclic order of the initial pieces of P_1, P_2, and P_3 around u is opposite to the cyclic order of their final pieces around v.

2.3 Theorem ([23]). *A graph is planar if and only if it has a drawing, in which every trifurcation is a converter.*

The second half of the theorem is trivial: if a graph is planar, then it can be drawn without crossing, and, clearly, every trifurcation in this drawing is a converter. The first half of the statement can be proved using Kuratowski's theorem.

Recently, G. Cairns and Y. Nikolayevsky [7] has improved the factor *two* in Theorem 2.1 to *one and a half.*

3 Different Crossing Numbers?

As is illustrated by Theorem 2.3, the investigation of crossings in graphs often requires parity arguments. This phenomenon can be partially explained by the 'banal' fact that if we start out from the interior of a simple (non-selfintersecting) closed curve in the plane, then we find ourselves inside or outside of the curve depending on whether we crossed it an even or an odd number of times.

Next we define three variants of the notion of crossing number.

(1) The *rectilinear crossing number*, $\text{LIN-CR}(G)$, of a graph G is the minimum number of crossings in a drawing of G, in which every edge is represented by a straight-line segment.

(2) The *pairwise crossing number* of G, $\text{PAIR-CR}(G)$, is the minimum number of crossing pairs of edges over all drawings of G. (Here the edges can be represented by arbitrary continuous curves, so that two edges may cross more than once, but every pair of edges can contribute to $\text{PAIR-CR}(G)$ at most one.)

(3) The *odd-crossing number* of G, $\text{ODD-CR}(G)$, is the minimum number of those pairs of edges which cross an odd number of times, over all drawings of G.

It readily follows from the definitions that

$$\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G).$$

Bienstock and Dean [6] exhibited a series of graphs with crossing number 4, whose rectilinear crossing numbers are arbitrary large. However, we cannot rule out the possibility that

$$\text{ODD-CR}(G) = \text{PAIR-CR}(G) = \text{CR}(G),$$

for every graph G.

The determination of the odd-crossing number can be rephrased as a purely combinatorial problem, thus the possible coincidence of the above three crossing numbers would offer a spark of hope that there exists an efficient approximation algorithm for computing their value.

According to a remarkable theorem of Hanani (alias Chojnacki) [8] and William Tutte [39], if a graph G can be drawn in the plane so that any pair of its edges cross an even number of times, then it can also be drawn without any crossing. In other words, $\text{ODD-CR}(G) = 0$ implies that $\text{CR}(G) = 0$. Note that in this case, by the observation of Fáry mentioned in Section 2, we also have that $\text{LIN-CR}(G) = 0$.

The main difficulty in this problem is that a graph has so many essentially different drawings that the computation of any of the above crossing numbers, for a graph of only 15 vertices, appears to be a hopelessly difficult task even for a very fast computer [10].

3.1 Theorem [12],[31]. *The computation of the crossing number, the pairwise crossing number, and the odd-crossing number are NP-complete problems.*

All we can show is that the three parameters in Theorem 3.1, $\textsc{cr}(G)$, $\textsc{pair-cr}(G)$, and $\textsc{odd-cr}(G)$, are not completely unrelated.

3.2 Theorem [31]. *For any graph G, we have*

$$\textsc{cr}(G) \leq 2(\textsc{odd-cr}(G))^2.$$

The proof of the last statement is based on the following sharpening of the Hanani–Tutte Theorem.

3.3 Theorem [31]. *An arbitrary drawing of any graph in the plane can be re-drawn in such a way that no edge, which originally crossed every other edge an even number of times, would participate in any crossing.*

In [28], we apply the original form of the Hanani–Tutte Theorem to answer a question raised in robotics [19].

4 Straight-line Drawings

For "straight-line thrackles," Conway's conjecture discussed in Section 2 had been settled by H. Hopf–E. Pannwitz [15] and (independently) by Paul Erdős much before the problem was raised.

If every edge of a graph is drawn by a straight-line segment, then we call the drawing a *geometric graph* [24], [25], [26]. Two geometric graphs are considered isomorphic (identical), if and only if there is a rigid motion of the plane which takes one into the other.

Hopf–Pannwitz–Erdős Theorem. *If any two edges of a geometric graph intersect (in an endpoint or an internal point), then it can have at most as many edges as vertices.*

The systematic study of extremal problems for geometric graphs was initiated by S. Avital–H. Hanani [4], Erdős, Micha Perles, and Yaakov Kupitz [21]. In particular, they asked the following question: what is the maximum number of edges of a geometric graph of n vertices, which does not have k pairwise disjoint edges? (Here, by "disjoint" we mean that they cannot cross and cannot even share an endpoint.) Denote this maximum by $e_k(n)$.

Using this notation, the above theorem says that $e_2(n) = n$, for every $n > 2$. Noga Alon and Erdős [2] proved that $e_3(n) \leq 6n$. Since then, this bound was reduced by a factor of two [13]. It had been an open problem for a long time to decide whether $e_k(n)$ is linear in n for every fixed $k > 3$.

4.1 Theorem [32]. *For every k and every n, we have $e_k(n) \leq (k-1)^4 n$.*

This bound was improved successively by Géza Tóth–Pavel Valtr [37], and by Tóth to $e_k(n) \leq 100k^2 n$. It is very likely that the dependence of $e_k(n)$ on k is also (roughly) linear.

Analogously, one can try to determine the maximum number of edges of a geometric graph with n vertices, which does not have k pairwise crossing edges. Denote this maximum by $f_k(n)$. It follows from Euler's Polyhedral Formula that, for $n > 2$, every planar graph with n vertices has at most $3n - 6$ edges. Equivalently, we have $f_2(n) = 3n - 6$.

4.2 Theorem [1]. $f_3(n) = O(n)$.

4.3 Theorem [27]. *For a fixed $k > 3$, we have $f_k(n) = O(n \log^{2k-6} n)$.*

Recently, Valtr [40] has shown that $f_k(n) = O(n \log n)$, for any $k > 3$, but it can be conjectured that $f_k(n) = O(n)$. Moreover, it cannot be ruled out that there exists a constant c such that $f_k(n) \leq ckn$, for every k and n. However, we cannot even decide whether every complete geometric graph with n vertices contains at least (a positive) constant times n pairwise crossing edges. The strongest result in this direction is the following

4.4 Theorem [3]. *Every complete geometric graph with n vertices contains at least $\lfloor\sqrt{n/12}\rfloor$ pairwise crossing edges.*

In a recent series of papers [16], [17], [18], we established some *Ramsey-type* results for geometric graphs, closely related to the subject of this section. In [9], we generalized the above results for *geometric hypergraphs* (systems of simplices).

5 An Application in Computer Graphics

It is a pleasure for the mathematician to see his research generate some interest outside his narrow field of studies. It is a source of even greater satisfaction if his results can be applied in other disciplines or, at some special and rare occasions, in practice.

During the past twenty years, combinatorial geometers have been fortunate enough to experience this feeling quite often. Automated production lines revolutionized *robotics*, and started an avalanche of questions whose solution required new combinatorial geometric tools [35]. *Computer graphics*, whose group of users encompasses virtually everybody from engineers to film-makers, has had a similar effect on our subject [5].

Finally, I would like to sketch a mathematical result which has applications in computer graphics. Most graphics packages available on the market contain some (so-called *warping* or *morphing*) program suitable for deforming figures or pictures. Originally, these programs were written for making commercials and animated movies, but today they are widely used.

An important step in programs of this type is to fix a few basic points of the original picture (say, the vertices of the straight-line drawing of a planar graph), and then to choose new locations for these points. We would like to re-draw the graph without creating any crossing. In general, we cannot now insist that the edges be represented by segments, because such a drawing may not exist. Our goal is to produce a drawing with polygonal edges, whose total number of segments is small. The complexity and the running time of the program is proportional to this number.

5.1 Theorem [33]. *Every planar graph with n vertices can be re-drawn in such a way that the new positions of the vertices are arbitrarily prescribed, and each edge is represented by a polygonal path consisting of at most $24n$ segments.*

There is an $O(n^2)$–time algorithm for constructing such a drawing.

The next result shows that Theorem 5.1 cannot be substantially improved.

5.2 Theorem [33]. *For every n, there exist a planar graph G_n with n vertices and an assignment of new locations for the vertices such that in any polygonal drawing of G_n there are at least $n/100$ edges composed of at least $n/100$ segments.*

The proof of this theorem is based on a result discovered by Leighton [22] (and slightly generalized in [27]), which turned out to play a crucial role in the solution of many other extremal and algorithmic problems related to graph embeddings.

The *bisection width* of a graph is the minimum number of edges whose removal splits the graph into two pieces such that there are no edges running between them and the larger piece has at most twice as many vertices as the smaller.

5.3 Theorem [22],[27]. *Let G be a graph of n vertices whose degrees are*

$d_1, d_2, \ldots, d_n$. *Then the bisection width of G is at most*

$$1.58\sqrt{16\textsc{cr}(G) + \sum_{i=1}^{n} d_i^2}.$$

References

[1] P. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir: Quasi-planar graphs have a linear number of edges, *Combinatorica* **17** (1997), 1–9.

[2] N. Alon and P. Erdős: Disjoint edges in geometric graphs, *Discrete and Computational Geometry* **4** (1989), 287-290.

[3] B. Aronov, P. Erdős, W. Goddard, D. J. Kleitman, M. Klugerman, J. Pach, and L. J. Schulman: Crossing families, *Combinatorica* **14** (1994), 127–134.

[4] S. Avital and H. Hanani: Graphs, *Gilyonot Lematematika* **3** (1966), 2-8 (in Hebrew).

[5] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf: *Computational Geometry – Algorithms and Applications*, Springer-Verlag, Berlin, Heidelberg, 1997.

[6] D. Bienstock and N. Dean: Bounds for rectilinear crossing numbers, *Journal of Graph Theory* **17** (1993), 333–348.

[7] G. Cairns and Y. Nikolayevsky: Bounds for generalized thrackles, *Discrete and Computational Geometry*, to appear.

[8] Ch. Chojnacki (A. Hanani): Über wesentlich unplättbare Kurven im dreidimensionalen Raume, *Fund. Math.* **23** (1934), 135–142.

[9] T. K. Dey and J. Pach: Extremal problems for geometric hypergraphs, *Discrete and Computational Geometry* **19** (1998), 473–484.

[10] P. Erdős and R. K. Guy: Crossing number problems, *American Mathematical Monthly* **80** (1973), 52–58.

[11] I. Fáry: On straight line representation of planar graphs, *Acta Univ. Szeged. Sect. Sci. Math.* **11** (1948), 229–233.

[12] M. R. Garey and D. S. Johnson: Crossing number is NP-complete, *SIAM Journal of Algebraic and Disccrete Methods* **4** (1983), 312–316.

[13] W. Goddard, M. Katchalski, and D. J. Kleitman: Forcing disjoint segments in the plane, *European Journal of Combinatorics* **17** (1996), 391-395.

[14] R. K. Guy: The decline and fall of Zarankiewicz's theorem, in: *Proof Techniques in Graph Theory*, Academic Press, New York, 1969, 63–69.

[15] H. Hopf and E. Pannwitz: Aufgabe No. 167, *Jahresbericht der Deutschen Mathematiker-Vereinigung* **43** (1934), 114.

[16] G. Károlyi, J. Pach, and G. Tóth: Ramsey-type results for geometric graphs. I, *Discrete and Computational Geometry* **18** (1997), 247–255.

[17] G. Károlyi, J. Pach, G. Tardos, and G. Tóth: An algorithm for finding many disjoint monochromatic edges in a complete 2-colored geometric graph, in: Intuitive Geometry (I. Bárány and K. Böröczky, eds.) Bolyai Society Mathematical Studies **6**, Budapest, 1997, 367–372.

[18] G. Károlyi, J. Pach, G. Tóth, and P. Valtr: Ramsey-type results for geometric graphs. II, *Discrete and Computational Geometry* **20** (1998), 375–388.

[19] K. Kedem, R. Livne, J. Pach, and M. Sharir: On the union of Jordan regions and collision–free translational motion amidst polygonal obstacles, *Discrete and Computational Geometry* **1** (1986), 59–71.

[20] D. J. Kleitman: The crossing number of $K_{5,n}$, *Journal of Combinatorial Theory* **9** (1970), 315–323.

[21] Y. Kupitz: Extremal problems in combinatorial geometry, *Aarhus University Lecture Notes Series* **53**, Aarhus University, Denmark, 1979.

[22] T. Leighton: *Complexity Issues in VLSI, Foundations of Computing Series,* MIT Press, Cambridge, MA, 1983.

[23] L. Lovász, J. Pach, and M. Szegedy: On Conway's thrackle conjecture, *Discrete and Computational Geometry* **18** (1997), 369–376.

[24] J. Pach: Notes on geometric graph theory, in: Discrete and Computational Geometry (J.E. Goodman et al., eds.), DIMACS Series, Vol 6, Amer. Math. Soc., Providence, 1991, 273–285.

[25] J. Pach: Geometric graphs and geometric hypergraphs, *Graph Theory Notes of New York* **31** (1996), 39–43.

[26] J. Pach and P.K. Agarwal: *Combinatorial Geometry*, J. Wiley and Sons, New York, 1995.

[27] J. Pach, F. Shahrokhi, and M. Szegedy: Applications of the crossing number, *Algorithmica* **16** (1996), 111–117.

[28] J. Pach and M. Sharir: On the boundary of the union of planar convex sets, *Discrete and Computational Geometry* (1998), to appear.

[29] J. Pach, J. Spencer, and G. Tóth: New bounds for crossing numbers, in preparation.

[30] J. Pach and G. Tóth: Graphs drawn with few crossings per edge, *Combinatorica* **17** (1997), 427–439.

[31] J. Pach and G. Tóth: Which crossing number is it, anyway?, in: FOCS'98, to appear. Also in: *Journal of Combinatorial Theory, Ser. B.*

[32] J. Pach and J. Törőcsik: Some geometric applications of Dilworth's theorem, *Discrete and Computational Geometry* **12** (1994), 1–7.

[33] J. Pach and R. Wenger: Embedding planar graphs with fixed vertex locations, in: Graph Drawing '98 (Sue Whitesides, ed.), Lecture Notes in Computer Science, Springer-Verlag, Berlin, 1999, to appear.

[34] R. B. Richter and C. Thomassen: Relations between crossing numbers of complete and complete bipartite graphs, *American Mathematical Monthly*, February 1997, 131–137.

[35] M. Sharir: Motion planning, in: *Handbook of Discrete and Computational Geometry*, (J. E. Goodman and J. O'Rourke, Eds.), CRC Press, Boca Raton, Florida, 1997, 733–754.

[36] G. Tóth: Geometric graphs with few disjoint edges II, in preparation.

[37] G. Tóth and P Valtr: Geometric graphs with few disjoint edges, in: Proc. 14th Annual Symp. on Computational Geometry, ACM Press, 1998, 184–191. Also in: *Discrete and Computational Geometry*, to appear.

[38] P. Turán: A note of welcome, *Journal of Graph Theory* **1** (1977), 7–9.

[39] W. T. Tutte: Toward a theory of crossing numbers, *Journal of Combinatorial Theory* **8** (1970), 45–53.

[40] P. Valtr: On geometric graphs with no k pairwise parallel edges, *Discrete and Computational Geometry* **19** (1998), 461–469.

[41] D. R. Woodall: Thrackles and deadlock, in: *Combinatorial Mathematics and Its Applications* (D.J.A. Welsh, ed.), Academic Press, London, 1969, 335–348.