A Long Noncrossing Path Among Disjoint Segments in the Plane*

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Abstract

Let $\mathcal{L}$ be a collection of n pairwise disjoint segments in general position in the plane. We show that one can find a subcollection of $\Omega(n^{1/3})$ segments that can be completed to a noncrossing simple path by adding rectilinear edges between endpoints of pairs of segments. On the other hand, there is a set $\mathcal{L}$ of n segments for which no subset of size $2n^{1/2}$ or more can be completed to such a path.

1 Introduction

Since the publication of the seminal paper of Erdős and Szekeres [3], many similar results have been discovered, establishing the existence of various regular subconfigurations in large geometric arrangements. The classical tool for proving such theorems is Ramsey theory [2]. However, the size of the regular substructures guaranteed by Ramsey's theorem are usually very small (at most logarithmic) in terms of the size n of the underlying arrangement. In most cases, the results are far from optimal. One can obtain better bounds (n^{ε} for some $\varepsilon > 0$) by introducing some linear orders on the elements of the arrangement and applying some Dilworth-type theorems [1] for partially ordered sets [9], [5], [8]. A simple one-dimensional prototype of such a statement is the Erdős-Szekeres lemma: any sequence of n real numbers has a monotone increasing or monotone decreasing subsequence of length $\lceil\sqrt{n}\rceil$. In this note, we give a new application of this idea.

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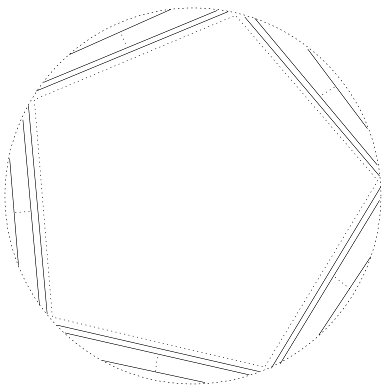

Figure 1: An arrangement of segments showing that $f(n) \le 2\sqrt{2n}$

A collection $\mathcal{L}$ of segments in the plane is in *general position* if no two elements of $\mathcal{L}$ are parallel, all of their endpoints are distinct, and no three endpoints are collinear. A polygonal path $P = p_1 p_2 \ldots p_n$ is called *simple* if no pair of its vertices coincide, i.e., $p_i \neq p_j$ whenever $i \neq j$. It is called *noncrossing* if no two edges share an interior point. A polygonal path P is called *alternating* with respect to $\mathcal{L}$ if every other edge of P belongs to $\mathcal{L}$.

We consider the following old problem of unknown origin: what is the maximum length $f(n)$ of an alternating path that can be found in any collection of n pairwise disjoint segments in the plane in general position? This question was included in a list of "Open problems in computational geometry" collected and annotated by Urrutia [11]. The easy construction described there can be slightly improved to show that $f(n) \le 2\sqrt{2}\sqrt{n}$ for $n = 2k^2$. Consider a $2k$-gon inscribed in a circle C and replace each of its edges e with k pairwise disjoint chords of C, almost parallel to e, that are farther away from the center of C than e is. (See Figure 1.) It seems likely that the order of magnitude of this bound is not far from optimal. For some similar problems, see [4], [6], [7], [10].

First we consider the special case when all segments cross the same line.

Theorem 1. *Let $\mathcal{L}$ be a collection of n pairwise disjoint segments in general position in the plane, all of whose members cross a given line. Then one can select $\Omega(n^{1/2})$ segments from $\mathcal{L}$ that can be completed to a noncrossing simple alternating path.*

The following result is an easy corollary of Theorem 1.

Theorem 2. *The maximum length $f(n)$ of an alternating path that can be found in any collection of n pairwise disjoint segments in the plane satisfies $f(n) = \Omega(n^{1/3})$.*

To see that the latter result follows from Theorem 1, observe that (e.g., by the Dilworth theorem) any collection $\mathcal{L}$ of n pairwise disjoint segments has a subcollection $\mathcal{L}_1$ consisting of least $n^{1/3}$ segments whose projections to the x-axis are pairwise disjoint, or a subcollection $\mathcal{L}_2$ consisting of at least $n^{2/3}$ segments, all of which can be crossed by a line parallel to the y-axis. In the first case, the elements of $\mathcal{L}_1$ can be connected to form an alternating path. In the second case, we can apply Theorem 1.

2 Proof of Theorem 1

Assume without loss of generality that all segments cross the y-axis, no two of them are parallel, and all $2n$ coordinates of their endpoints are distinct. The above-below relation between the crossings of the segments with the y-axis induces a natural linear order on the elements of $\mathcal{L}$. We apply the Erdős-Szekeres lemma to find a subsequence of $\mathcal{L}$ consisting of $\lceil\sqrt{n}\rceil$ segments with increasing or decreasing slopes with respect to this order. Since we can always flip the plane about the y-axis, we may assume that the slopes of the elements of this subsequence are monotone increasing. In what follows, for convenience we assume that $\sqrt{n}$ and all other numbers that appear in the argument (except the coordinates of the endpoints) are integers satisfying the necessary divisibility conditions so that we do not have to use "floor" and "ceiling" operations. This will not effect the asymptotic results obtained in this paper.

To be more precise, we find a sequence of at least $\sqrt{n}$ segments $s_1, \ldots, s_m$ ($m = \sqrt{n}$) of $\mathcal{L}$ such that if $i < j$, then s_i is above s_j and the slope of s_i is smaller than that of s_j (see Figure 2).

Partition $s_1, \ldots, s_m$ into $k = m/5$ groups, each consisting of 5 consecutive segments. That is, let $G_i = \{s_{5(i-1)+1}, \ldots, s_{5(i-1)+5}\}$ for every $1 \le i \le k$. For each G_i, apply again the Erdős-Szekeres lemma and find a subsequence of 3 segments such that the x-coordinates of their right endpoints form an increasing or a decreasing sequence. By flipping the plane about the x-axis, if necessary, we can also assume that for at least half of the G_is, these sequences are decreasing. From now on, we disregard all other segments. Summarizing: we now have $k/2$ groups $L_1, \ldots, L_{k/2}$, each consisting of 3 elements of $\mathcal{L}$. For each $1 \le i \le k/2$, let $L_i = \{\ell_1^i, \ell_2^i, \ell_3^i\}$, where ℓ_b^a is above $\ell_{b'}^{a'}$ and its slope is smaller, whenever $a < a'$, or if $a = a'$ and $b < b'$. Moreover, for a fixed a and any $b < b'$, the x-coordinate of the right endpoint of ℓ_b^a is larger than that of $\ell_{b'}^a$. Let $\mathcal{S} := L_1 \cup \ldots \cup L_{k/2}$.

Denote by p_b^a and q_b^a the left endpoint and the right endpoint of ℓ_b^a, respectively. For any two points r, s, let $[r, s]$ stand for the segment connecting r and s.

Define a set of auxiliary segments as follows. For $1 \le a \le k/2$ and $b = 1, 2$, let $z_b^a = [q_b^a, q_{b+1}^a]$. We say that z_b^a is *bad*, if there is a segment in $\mathcal{S}$ that meets the interior of z_b^a. For any segment $\ell_j^t \in \mathcal{S}$ meeting the interior of z_b^a, we have $t > a$, because all elements of $\cup_{t<a} L_t$ lie strictly above z_b^a, otherwise they would cross ℓ_b^a. Define the *witness index* of a bad segment z_b^a as the smallest index $t > a$ with the property that there exists an ℓ_j^t meeting the interior of z_b^a.

Lemma 2.1. *If the witness index of a bad segment z_b^a is t, then ℓ_1^t meets z_b^a. Moreover, q_1^t must belong to the interior of the region enclosed by the y-axis, ℓ_b^a, ℓ_{b+1}^a, and z_b^a.*

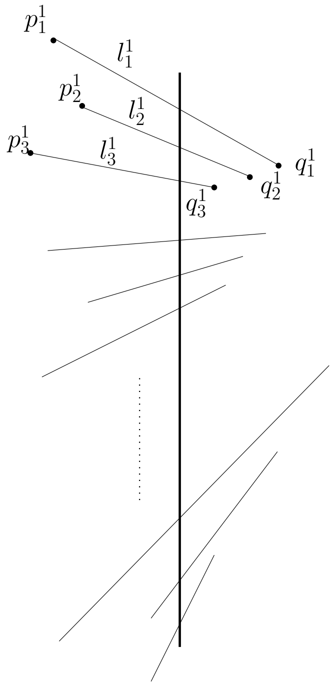

Figure 2: The segments l_1^i, l_2^i, l_3^i

Proof. We know that $t > a$ and that for some j the segment ℓ_j^t crosses z_b^a. Assume that $j > 1$. Let W denote the region bounded by the y-axis, ℓ_{b+1}^a, z_b^a, and ℓ_j^t. The segment ℓ_1^t lies above ℓ_j^t, and the x-coordinate of its right endpoint q_1^t is larger than the x-coordinate of q_j^t. Clearly, the intersection point r of ℓ_j^t and z_b^a is the rightmost corner of the boundary of W. There is a point on ℓ_1^t whose x-coordinate is the same as that of r. This point must lie above r and outside the region W. Since ℓ_1^t crosses the y-axis above ℓ_j^t and below ℓ_{b+1}^a, at a boundary point of R, and it has a point outside W, it must have another crossing with the boundary of W. Using the fact that the elements of $\mathcal{S}$ are pairwise disjoint, this second crossing must belong to z_b^a.

As for the second part of the lemma, let R denote the region bounded by the y-axis, ℓ_b^a, ℓ_{b+1}^a, and z_b^a. We have seen that ℓ_1^t meets the boundary of R (at a point of z_b^a). Since ℓ_1^t is disjoint from both ℓ_b^a and ℓ_{b+1}^a, and it intersects the y-axis below ℓ_{b+1}^a, it follows that ℓ_1^t cannot cross the boundary of R a second time. Therefore, q_1^t must belong to the interior of R. $\square$

Lemma 2.2. *No two different bad segments can have the same witness index.*

Proof. Assume to the contrary that t is the witness index of two bad segments, z_b^a and $z_{b'}^{a'}$. Suppose without loss of generality that ℓ_b^a lies above $\ell_{b'}^{a'}$. We know that both of them lie above ℓ_1^t. As in the proof of Lemma 2.1, let R denote the region bounded by the y-axis, ℓ_b^a, ℓ_{b+1}^a, and z_b^a. Similarly, let R' denote the region bounded by the y-axis, $\ell_{b'}^{a'}$, $\ell_{b'+1}^{a'}$, and $z_{b'}^{a'}$. R and R' do not overlap. Indeed, since the elements of

$\mathcal{S}$ are pairwise disjoint, R and R' could overlap only if $\ell_{b'}^{a'}$ crossed z_b^a. However, this would contradict the minimality of t.

On the other hand, by Lemma 2.1, ℓ_1^t must intersect both z_b^a and $z_{b'}^{a'}$, and its right endpoint q_1^t must belong to the interiors of both R and R'. We thus obtained the desired contradiction. $\square$

Now we are in a position to prove Theorem 1.

By Lemma 2.2, the number of bad segments is at most $k/2$. We say that an index i ($1 \leq i \leq k/2$) is *good* if at least one of the segments z_1^i, z_2^i is not bad. Obviously, at least $k/2 - \frac{k/2}{2} = k/4$ indices between 1 and $k/2$ are good. Assume without loss of generality that the first $k/4$ indices are good. To complete the proof it is sufficient to show how to draw a noncrossing simple alternating path P that uses the segments ℓ_2^i, ℓ_3^i (and perhaps even ℓ_1^i) for $1 \leq i \leq k/4 = \Omega(\sqrt{n})$.

Let the first points of P be $q_1^1, p_1^1, q_2^1, p_2^1, q_3^1, p_3^1$, in this order. That is, so far we have built a "zigzag" path that uses the segments $\ell_1^1, \ell_2^1, \ell_3^1$. Since 2 is a good index, there exists a segment z_j^2 ($j = 1$ or 2) which is not bad. Let us extend P by adding the vertices p_j^2, q_j^2, q_{j+1}^2, and hence adding the edges ℓ_j^2 (from left to right) and z_j^2. Next we can add the point p_{j+1}^2 and, if $j = 1$, also the points q_3^2, p_3^2, zigzagging just like before. Continuing in the same manner, we build a path P using at least two edges from each group L_i ($i \leq k/4$). It is easy to check that P is a *noncrossing* path, because (1) its edges belonging to $\mathcal{L} \subset \mathcal{S}$ are pairwise disjoint; (2) its edges to the left of the y-axis do not cross any other edge, by the assumption that the slopes of the elements of $\mathcal{S}$ form an increasing sequence; (3) its edges to the right of the y-axis are not bad, therefore they do not cross any other edge of P. This completes the proof of Theorem 1.

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