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Decomposition of multiple packing and covering

by

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In this talk I would like to call attention to a branch of problems which seem to appear collaterally in combinatorics and discrete geometry.

Let X be an underlying set and let $\underline{S}$ be a family of subsets of X. Further, let k be a cardinal. The set system $\underline{S}$ is said to form a k-fold packing if each element $x \in X$ is contained in at most k sets belonging to $\underline{S}$. /In particular, $\underline{S}$ forms a 1-fold packing if and only if the sets of $\underline{S}$ are pairwise disjoint./ Analogously, $\underline{S}$ is a k-fold covering if each element $x \in X$ is contained in at least k sets in $\underline{S}$. A 1-fold packing and a 1-fold covering are called simply packing, and covering, resp. Now, the general /set theoretical/ formulation of the problem sounds as follows:

PROBLEM A. Let $\underline{S}$ be a k-fold packing on X. What is the minimum number t for which the sets belonging to $\underline{S}$ can be divided into t classes $\underline{S}_1, \underline{S}_2, \ldots, \underline{S}_t$ such that each $\underline{S}_i$ forms a packing.

PROBLEM B. Let $\underline{S}$ be a k-fold covering on X. What is the maximum number t' for which the sets belonging to $\underline{S}$ can

be divided into t' classes $\underline{S}_1', \underline{S}_2', \ldots, \underline{S}_{t'}'$, such that each $\underline{S}_i'$ forms a covering.

If the underlying set X is finite and the family $\underline{S}$ satisfies various additional restrictions /e.g. the size of the members is limited, no j members of $\underline{S}$ are pairwise disjoint, etc./ then we obtain a large scale of combinatorial versions of the above two problems. If X is the d-dimensional Euclidean space and $\underline{S}$ consists of convex sets, balls or cubes whose position is subject to geometric limitations /they are placed parallelly, they are congruent, etc./ then a number of geometric problems arises. The investigation of these latter variants of Problems A and B was initiated by László Fejes Tóth. At this point I should like to emphasize that in this case the combinatorial and geometric ways of questioning are not only analogous but they are closely related and often consonant in the sense that they lead to similar difficulties. As the main line of this meeting is discrete geometry, I shall deal mostly with the geometric versions of the problem. For the very reason I will omit all geometric details /which are mainly of technical character/ and I will stress those simple combinatorial concepts which seem to be fruitful in solving the geometric questions. I hope to suggest that the combinatorial-

graph theoretical approach may be useful in the investigation of many discrete geometry problems of the above type. Most of the questions which I shall mention in the course of my talk are partly or completely unsolved. Anyway, I have to remark that I would be very glad to see at least some of these problems solved, without respect to whether or not their proof uses combinatorial tools.

1. Packing of convex sets

Consider first the 1-dimensional case. Let E^1 be the Euclidean straight line, k a fixed natural number, and let $\underline{S}$ be a system of intervals which form a k-fold packing. As far as I know Hajós /1957/ was the first who observed that the intervals of $\underline{S}$ can be divided into k classes so that no two intervals belonging to the same class intersect. /That is, every class forms a packing./ This result is clearly the best possible, and it is the prototype of Problem A.

Passing over to higher dimensions, all versions of Problem A become much more difficult. Since these difficulties are present already in the 2-dimensional case, we restrict our discussion to the convex sets of the plane. For the sake of simplicity all convex sets /discs/ are assumed to be open The first difficulty which arises, can be demonstrated by the following example: Consider n

straight lines $e_1, e_2, \ldots, e_n$ of the plane, in general position. Suppose that the minimal distance between their intersection points is at least ε. Let C be a sufficiently large circle containing all intersection points, and let S_i denote the strip of width $\frac{\varepsilon}{5}$, drawn around e_i. It is obvious that the sets $S_i \cap C$ /i=1,2,...,n/ form a 2-fold packing in the plane. On the other hand, no two of them are disjoint. This example shows that, without any further restriction on the set systems, it is not true that there exists a function n(k) such that every k-fold packing of convex, bounded discs is decomposable into $\leq n(k)$ packings. Before we turn to the investigation of the above question under additional limitations, we reformulate Problem A in terms of graph theory.

Consider a family $\underline{S} = \{C_1, C_2, C_3, \ldots\}$ of convex sets on the plane. The <u>representative graph</u> of $\underline{S}$ is defined to be a graph G_S whose each vertex x_i corresponds to a convex set C_i, and in which two vertices are joined together if and only if the corresponding sets intersect. Now, Problem A is equivalent to

PROBLEM A'. Let $\underline{S}$ be a k-fold packing of convex sets on the plane, and let $G_{\underline{S}}$ denote the representative graph of $\underline{S}$. What is the chromatic number $\chi(G_{\underline{S}})$ of $G_{\underline{S}}$?

As has already been noted, no upper bound depending only on k can be given for $\chi(G_{\underline{S}})$. Our above example suggests that the chromatic number of the representative graph can be high only in that case when the family $\underline{S}$ contains "longish" sets. Theorem 1 shows that this is in fact true.

Let C be any convex set. Let r(C) and a(C) denote the radius of the circumscribed circle and the area of C, resp. Then the quantity $\ell(C) := \frac{r^2(C)\pi}{a(C)}$ expresses something like how longish C is.

THEOREM 1. Let $\underline{S} = \{C_1, C_2, C_3, \ldots\}$ be a k-fold packing of convex sets of the plane. Suppose that $\ell(C_i) \leq L$ for all $i=1,2,3,\ldots$. Then $\underline{S}$ is decomposable into $\leq 9Lk$ packings.

PROOF /outline/. Let $\underline{S}'$ be any finite subfamily of $\underline{S}$. Let C denote that member of $\underline{S}'$ for which r(C) is minimum. Easy geometric arguments show that C intersects at most 9Lk-1 members of $\underline{S}'$, which means that the vertex x corresponding to C has at most 9Lk-1 neighbours in the representative graph $G_{\underline{S}'}$. In other words, every finite subgraph of $G_{\underline{S}}$ has a vertex of degree $\leq 9Lk-1$. From here, by induction, every finite subgraph of $G_{\underline{S}}$ is 9Lk-colourable. Consequently, using the deBruijn-Erdős Compactness Theorem, $G_{\underline{S}}$ is also 9Lk-colourable, $\chi(G_{\underline{S}}) \leq 9Lk$. ∎

Remark that the condition of Theorem 1 is automatically satisfied /with a suitable L/ in that particular case when $\underline{S}$ consists of similar copies of a fixed convex set. For instance, if $\underline{S}$ is a k-fold packing of circles, then our theorem states $\chi(G_{\underline{S}}) \leq 9k$. /By means of simple geometric tricks, this bound can be essentially improved. E.g. if the circles are equal, then instead of 9k, 5k can be written./

THEOREM 2. Let $\underline{S}$ be a 2-fold packing on the plane, which consists of homothetic /similar, translated/ copies of a convex disc C. Then

(i) $\underline{S}$ is decomposable into 4 packings.

(ii) If C is a parallelogramm, then $\underline{S}$ is decomposable into 3 packings.

PROOF /outline/. It is easily verified that in this case the representative graph $G_{\underline{S}}$ is planar. It now follows from the Four Colour Theorem that $\chi(G_{\underline{S}}) \leq 4$. If C is a parallelogramm, then $G_{\underline{S}}$ is, in addition, triangle free. Grötzsch' theorem asserts that every triangle free, planar graph is 3-colourable. □

We conclude this part by listing some open problems in this area.

PROBLEMS.

(i) Determine the least number $n(k)$ $(\leq 5k)$ for which every k-fold packing of equal circles can be decomposed into $n(k)$ packings. /L. Fejes Tóth./

(ii) Is it true that every planar graph is the representative graph of a 2-fold circle packing of (not necessarily equal) circles ? Or a weaker version of the same question: Is it true that every planar graph is the representative graph of an arrangement of circles?

(iii) Characterize all graphs which are representative graph of some arrangement of equal circles.

(iv) Prove (i) of Theorem 2 without use of the Four Colour Theorem.

In the spirit of Problem (iii) we introduce the following notion. The dimension of a finite graph G is defined to be the smallest number d for which G is embeddable in the d-dimensional Euclidean space so that two points are joined by an edge if and only if their distance is ≤ 1 . /This definition is correct, since it is not difficult to prove that every graph with n vertices is embeddable in E^{n-1}, in the desired way./ Another /equivalent/ form of Problem (iii) is: Characterize the 2-dimensional graphs. I think that the investigation of

the dimension of graphs may lead to interesting consequences in both graph theory and discrete geometry. We mention only one of the numerous arising questions lying in the borderland of these two fields.

(v) Give a good upper bound for the maximal dimension of a graph having n vertices.

2. Covering with convex sets

Let $\underline{S}$ be a system of intervals which form a k-fold covering of E^1. Then the members of $\underline{S}$ can be divided into k classes so that each class covers the whole line. In other words, $\underline{S}$ is decomposable into k coverings. The higher-dimensional versions of this problem /Problem A/ are far from being so simple.

Let k be any fixed natural number. Somewhat surprisingly, it is not difficult to construct a k-fold covering $\underline{S}_k$ of the plane, with convex open sets, such that $\underline{S}_k$ is not decomposable into two coverings. Consider $\binom{2k-1}{k}$ distinct points lying on a circle arc. Set up a one-to-one correspondance between these points and the unordered k-tuples of the first 2k-1 natural numbers. Let T_i denote the convex hull of those points, the k-tuples corresponding to which contain i /i=1,2,...,2k-1/. Blowing up slightly the sets T_i, we obtain a family of open sets $\{T_1', T_2', \ldots, T_{2k-1}'\}$, which can be completed to a k-fold covering $\underline{S}_k$ of the

whole plane, so that each of the $\binom{2k-1}{k}$ points is covered exactly k times. It is easy to see that $\underline{S}_k$ is not decomposable into two coverings.

The simplest unsolved question is the following.

PROBLEM (vi). Is it true that, for a sufficiently large k, every k-fold covering of the plane with equal circles is decomposable into two coverings?

PROPOSITION 1. Let $\underline{S}$ be a system of open sets in the plane. $\underline{S}$ is decomposable into t coverings of the plane, if and only if, for every circle C, there is a finite subsystem $\underline{S}'$ of $\underline{S}$ which is decomposable into t coverings of C.

In view of this observation it is enough to concentrate our attention on finite set systems.

Let $\underline{S} = \{S_1, S_2, \ldots, S_n\}$ be a finite family of open sets which form a k-fold covering of a circle C. Then $\underline{S}$ defines a partition of C into finitely many classes $C_1, C_2, \ldots, C_m$, in a natural way: Two points belong to the same class if and only if they are contained in the same members of $\underline{S}$. The <u>representative hypergraph</u> $H_{\underline{S}}$ of $\underline{S}$ is defined to be a hypergraph, whose vertices $x_1, x_2, \ldots, x_n$ and hyperedges $E_1, E_2, \ldots, E_m$ correspond to the members of $\underline{S}$ and the classes of C, resp. More precisely, $x_i \in E_j \iff S_i \supseteq C_j$.

Now, the question whether a covering $\underline{S}$ of C is decomposable into t coverings, can be reformulated as follows: Are the vertices of $H_{\underline{S}}$ colourable with t colours so that each colour occurs in each hyperedge?

The following theorems give a partial answer to Problem (vi).

THEOREM 3. Let $\underline{S}$ be a k-fold covering of the plane with equal circles. Suppose, in addition, that $\underline{S}$ forms an $\leq 2^{\frac{k-11}{2}}$-fold packing. Then $\underline{S}$ is decomposable into two coverings.

THEOREM 4. There is a constant c, such that every k-fold lattice-covering with circles is decomposable into $c\,\frac{k}{\log k}$ coverings.