The Wave Equation

Ignoring boundary conditions:

$$u_{tt} = c^2 u_{xx}$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

The solution can be given explicitly via d'Alembert's Formula:

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

The Domain of Dependence at X, T is then all $x \in [x-ct, x+ct]$ due to the integral term

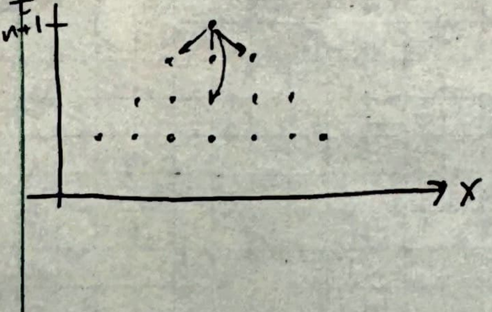
Simple Finite diff approach: centered differences:

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = c^2 \left(\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2} \right) + O(k^2)$$

$$\Rightarrow \quad U_j^{n+1} = 2U_j^n - U_j^{n-1} + \frac{c^2 k^2}{h^2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n \right)$$

stability will generally require $\left|\frac{ck}{h}\right| < 1$.

Numerical domain of dependence:



To initialize the scheme use the exact solution:

$$u(x,k) = \frac{f(x-ck) + f(x+ck)}{2} + \frac{1}{2c}\int_{x-ck}^{x+ck} g(s)\, ds$$

If only grid values are known, interpolate such that error in $u(x,k)$ is $O(k^3)$.