Last Time

GMRES: Generate $\mathcal{K}_n = \{\vec{b}, \ldots, A^{n-1}\vec{b}\}$, search for $\vec{x}^{(n)} \in \mathcal{K}_n$ to minimize $\| A\vec{x}^{(n)} - \vec{b} \|_2$.

To do this efficiently and accurately we use Arnoldi iteration to build a matrix Q_n such that $Q_n^T Q_n = I$ and $\mathrm{col}(Q_n) = \mathcal{K}_n$. (Hessenberg form, etc.)

Alternative interpretation: polynomial approximation.

$$\vec{r}^{(n)} = \vec{b} - A\vec{x}^{(n)}$$
$$= \vec{b} - A(\vec{b} \; A\vec{b} \cdots A^{n-1}\vec{b})\vec{c}$$
$$= (I - (c_1 A\vec{b} + \ldots + c_n A^n \vec{b}))$$
$$= p(A)\vec{b}, \quad p(0) = 1.$$

so $\| \vec{r}^{(n)} \|_2 \le \| p(A) \| \cdot \| \vec{b} \|$

↳ make this small on the spectrum of A.

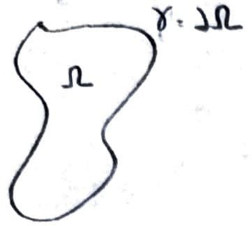
Where are we:
- Fourier analysis, spectral approximation
- 1D BVPs
  - spectral methods, diff vs. Int.
  - finite difference. → motivated 2D/3D elliptic problems

elliptic equations.

- Iterative solvers + sparse matrices.

Next: Overview of Integral equations.

Canonical Problem:



$$\left.\begin{array}{ll} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \gamma \end{array}\right\} \quad \text{Laplace. Interior Dirichlet problem}$$

F.D. $\Rightarrow$

- geometry poses a problem
- $O(N^2)$ points needed to discretize the problem
- $O(N)$ ~~pts~~ points on γ

__Idea:__ Reformulate the PDE as a __boundary__ __integral__ __equation__ using the Greens function.

$$\int_0^{2\pi}\int_0^{R} \nabla\cdot\nabla G \, r \, dr\, d\theta = \int_{\partial B_R} \nabla G \cdot \hat{r} \, ds = \int \frac{1}{2\pi r} \hat{r}\cdot\hat{r} \, ds = \int_{\partial B_R} \frac{1}{2\pi r} ds = \int_0^{2\pi} \frac{1}{2\pi R} R\, d\theta = 1$$

In 2D: $\quad \Delta G(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y})$

$$\Rightarrow \quad G(\vec{x},\vec{y}) = \frac{1}{2\pi}\log r = \frac{1}{2\pi}\log|\vec{x}-\vec{y}|$$

$G(\vec{x},\vec{y})$ satisfies $\Delta G = 0$ so long as $\vec{x} \neq \vec{y}$

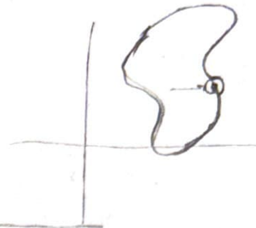
Green's Third Identity: If u is harmonic ($\Delta u = 0$) in Ω, then

$$(*) \quad u(\vec{x}) = -\underbrace{\int_\gamma \frac{\partial G}{\partial n_y}(\vec{x},\vec{y})\, u(\vec{y})\, ds(\vec{y})}_{\text{Double layer kernel } \mathcal{D}[u](\vec{x})} + \underbrace{\int_\gamma G(\vec{x},\vec{y}) \frac{\partial u}{\partial n_y}(y)\, ds(\vec{y})}_{\text{single layer kernel } = \mathcal{S}\left[\frac{\partial u}{\partial n_y}\right](\vec{x})} \quad \text{for } x \in \Omega$$

If we __know__ $\frac{\partial u}{\partial n_y}$ on the boundary γ, then we have a formula for u in Ω. Write down equation for $\frac{\partial u}{\partial n_y}$. We need $(*)$ for $\vec{x}$ on the boundary.

One can derive the following ~~matrix~~ limits by using Calc III/Complex (see Intro to PDE, Colton)

$$\lim_{\vec{x}\to\vec{x}_0} \mathcal{D}[\sigma](\vec{x}) = -\frac{1}{2}\sigma(\vec{x}_0) + \mathcal{D}[\sigma](\vec{x}_0)$$



$$\lim_{\vec{x}\to\vec{x}_0} \mathcal{S}[\sigma](\vec{x}) = \mathcal{S}[\sigma](\vec{x}_0)$$

$$\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

$$\frac{\partial}{\partial n_y}\frac{1}{2\pi}\log|\vec{x}-\vec{y}| = \frac{1}{2\pi}\hat{n}_y\cdot\nabla_y \log|\vec{x}-\vec{y}|$$

$$= \frac{1}{2\pi}\hat{n}_y\cdot\frac{1}{|\vec{x}-\vec{y}|}\frac{1}{2}\cdot 2\cdot\begin{pmatrix}-(x_1-y_1)\\ -(x_2-y_2)\end{pmatrix}\frac{1}{|\vec{x}-\vec{y}|} = \underbrace{-\frac{1}{2\pi}\frac{\hat{n}_y\cdot(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^2}}_{d(\vec{x},\vec{y})}$$

Test "jump relation"

$$\begin{aligned}\Delta u &= 0\\ u &= 1\end{aligned} \quad\Rightarrow\quad u = 1 \text{ in } \Omega$$

Compute $\mathcal{D}[1](\vec{x})$ for
- $\vec{x} \in \Omega$ (1)
- $\vec{x} \in \Gamma$ (.5)
- $\vec{x}$ outside Ω (0)

so on Γ Green's 3rd Identity turns into

$$u(\vec{x}) = \frac{1}{2}u(\vec{x}) - \mathcal{D}[u](\vec{x}) + \mathcal{S}\left[\frac{\partial u}{\partial n}\right](\vec{x}) \qquad \vec{x}\in\Gamma$$

$$\Rightarrow \underbrace{\mathcal{S}\left[\frac{\partial u}{\partial n}\right]}_{\text{solve}} = \underbrace{\frac{1}{2}u + \mathcal{D}[u]}_{\text{compute}}$$

First kind integral equation

$$\frac{\partial u}{\partial n} = \underbrace{\mathcal{S}^{-1}\left(\frac{1}{2} + \mathcal{D}\right)}_{\text{Dirichlet-to-Neumann map}}[u]$$

Dirichlet-to-Neumann map : $u \to \frac{\partial u}{\partial n}$

$\mathcal{S}$ has decaying spectrum. "slightly" ill-conditioned, $O(N) = \kappa$

An alternative integral equation:

Indirect BIE: Represent u as $u(\vec{x}) = \mathcal{D}[\sigma](\vec{x})$, σ defined on γ

- u satisfies $\Delta u = 0$ automatically
- enforcing the B.C. gives:

$$\lim_{\vec{x}' \to \vec{x}} \mathcal{D}[\sigma](\vec{x}') = f(\vec{x})$$

$$\Rightarrow \quad -\frac{1}{2}\sigma + \mathcal{D}[\sigma] = f \quad \text{on } \gamma \quad \Big] \; 2^{nd} \text{ Kind Int. Eq.}$$

Discretization Methods

- Collocation: $\sigma(s) = \sum c_i \varphi_i(s)$, enforce at $\vec{x}_i$

  Need "spectral" action of $\mathcal{D}$ on φ_i.

  $\Rightarrow$ yields linear system:

$$-\frac{1}{2}\sum_j c_j \varphi_j(t_i) + \sum_j \int_{\partial\gamma} d(x(t_i), y(s))\, c_j \varphi_j(s)\, ds = f(t_i)$$

- Galerkin: take inner products of ⊃ with another set of functions ψ_i

- Nystrom: Pick nodes $\vec{x}_i$ on Γ, and directly approximate the integral using a quadrature:

$$\Rightarrow \quad -\frac{1}{2}\sigma(\vec{x}_i) + \sum d(\vec{x}_i, \vec{x}_j)\, \sigma(\vec{x}_j) \underbrace{w_{ij}}_{\text{quadrature weights}} = f(\vec{x}_j)$$

All methods lead to <u>dense</u> linear systems.

- Colloc/Galerkin matrix entries are Integrals
- Nystrom matrix entries are function values $d(x_i, x_j)$.