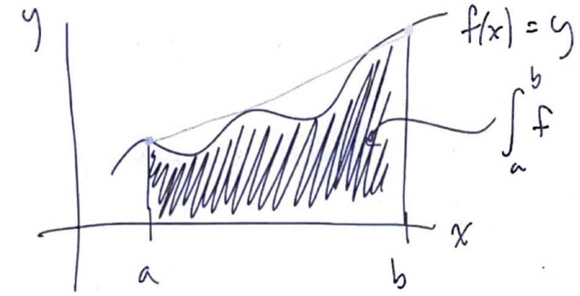
# The Trapezoidal (Trapezium) Rule

The most basic numerical integration technique:




Option 1: Approximate f on $[a,b]$ by a line, integrate the line.

Area under this line is $\underbrace{(b-a)}_{\text{width}} \underbrace{\frac{(f(a)+f(b))}{2}}_{\text{average of edge heights.}} \approx \int_a^b f(x)\,dx.$

Area of a trapezoid.

① f was approximated by a line which interpolated f at $x=a$ and $x=b$.

② This interpolating line was integrated.

The trapezoidal rule is a special case of more general numerical integration formulas known as Newton-Cotes rules:

<u>Idea</u> Interpolate f on $[a,b]$, and then integrate the interpolating polynomial.

This can be done with arbitrarily high degree (high order interpolation) but remember: the interpolating polynomial may suffer from <u>Runge's Phenomena</u>.

In general: Interpolate $f$ at the points $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$

$$p_n(x) = \sum_{j=0}^{n} L_j(x)\, f(x_j)$$

Lagrange function $L_j(x) = \prod_{i \neq j} \frac{x - x_i}{x_j - x_i}$

Then the integral of $f$ can be approximated as:

$$\int_a^b f(x)\,dx \approx \int_a^b p_n(x)\,dx$$

$$= \int_a^b \sum_{j=0}^{n} L_j(x)\, f(x_j)\,dx$$

$$= \sum_{j=0}^{n} \underbrace{\left( \int_a^b L_j(x)\,dx \right)}_{w_j} f(x_j)$$

(Numerical)

$$= \sum_{j=0}^{n} w_j\, f(x_j) \quad \Big\} \text{ Standard form of a \underline{Quadrature Rule}}$$

$w_j$ = quadrature weights

$x_j$ = quadrature nodes (the locations where $f$ is evaluated).

<u>Note</u>: $w_j = \int_a^b L_j(x)\,dx$ only depends on the location of the quadrature nodes, <u>not</u> the values of $f$.

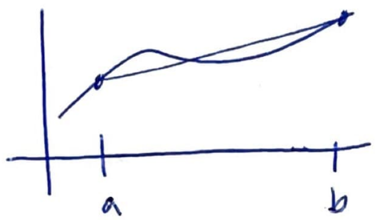
<u>Note 2</u>: Warning: Large $n$ may result in an inaccurate approximation by the interpolating polynomial due to Runge's Phenomena.

<u>Remedy</u>: Use many smaller lower-order interpolations, glue the results together.

# Honors Numerical Analysis

**Lecture 20**

Error in the Trapezoidal Rule:




The interpolant $p_1$ is given by

$$p_1(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}$$

The error for this interpolant is given from an analysis of Taylors Thm (from earlier):

$$f(x) - p_1(x) = \frac{1}{2!} f''(\xi_x)(x-a)(x-b)$$

$\xi_x$: some value in $[a,b]$

Let $\eta = \underset{x}{\operatorname{argmax}} |f''(\cancel{x})|$

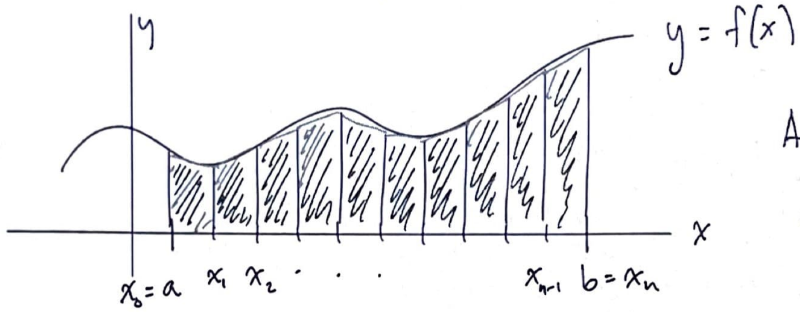
Then

$$\left| \int_a^b f(x) - \int_a^b p_1(x) \right| = \left| \int_a^b \frac{1}{2!} f''(\xi_x)(x-a)(x-b)\,dx \right|$$

$$\leq \frac{1}{2} f''(\eta) \left| \int_a^b (x-a)(x-b)\,dx \right|$$

$$= \frac{1}{12} f''(\eta) \underbrace{(b-a)^3}_{\text{call this } h^3,\ h = b-a}.$$

# The Composite Trapezoidal Rule




Apply the trapezoidal rule on each smaller interval (analogous to the Riemannian Integral).

$$x_j = a + jh$$
$$= a + j\left(\frac{b-a}{n}\right)$$

Then, the integral is approximated by:

$$\int_a^b f(x)\,dx \approx T_n f \quad \text{N-interval Trapezoidal Rule}$$
$$= \sum_{j=1}^{n} h\left(\frac{f(x_{j-1}) + f(x_j)}{2}\right)$$
$$= h\left(\sum_{j=0}^{n} f(x_j) - \frac{1}{2}\left(f(a) + f(b)\right)\right)$$

Question: What is the error $\left| \int_a^b f(x)\,dx - T_n f \right|$ ?

This error can be shown to be $O(h^2)$.

Idea of the proof: on a single interval $[a,b]$, Taylor expand $f$ about $x = \frac{a+b}{2}$ (the midpoint). Apply the Trapezoidal rule to this Taylor series.

In fact, a much more powerful formula exists that characterizes the error explicitly.

# The Euler-Maclaurin Expansion

**Theorem:** Let $f \in C^{2k}[a,b]$, and $[a,b]$ be divided into $n$ equal subintervals, $[x_{j-1}, x_j]$, with $x_j = a + jh$.

Then, $\int_a^b f(x)\,dx - T_n f$

$I$

$$= \sum_{r=1}^{k} c_r h^{2r} \left( f^{(2r-1)}(b) - f^{(2r-1)}(a) \right) - d_{2k} \left(\frac{h}{2}\right)^{2k}.$$

$$= \frac{h^2}{12}\left( f'(b) - f'(a) \right) - \frac{h^4}{720}\left( f^{(3)}(b) - f^{(3)}(a) \right) + \dots + (-1)^{k-1} \frac{B_{2k}}{(2k)!} h^{2k} f^{(2k)}(\xi)$$

$\xi \in [a,b]$

These coefficients are given by:

$$c_r = -\frac{B_{2r}}{(2r)!}, \quad B_{2r} \text{ is a Bernoulli number:}$$

$$\frac{x}{2}\coth\left(\frac{x}{2}\right) = \sum_{r=0}^{\infty} \frac{B_{2r}}{(2r)!} x^{2r}.$$

The calculation of $B_{2r}$ was the output of arguably the first "computer program" written by Ada Lovelace and Charles Babbage.

## Implications of Euler-Maclaurin

If $f \in C^{\infty}[a,b]$ and periodic with $f^{(j)}(a) = f^{(j)}(b)$ (think Fourier Series, or $\cos mx$, $\sin mx$, etc.) then this error $|I - T_n f|$ decays superalgebraically as $n \to \infty$.

Def: $\epsilon_n \to 0$ superalgebraically if

$$\lim_{n\to\infty} \frac{\epsilon_n}{h^{p \cdot n}} = 0 \quad \text{for any } p>0.$$

This means that $\epsilon_n \to 0$ faster than any power of $h$.

For this reason, the trapezoidal rule is very important important in various numerical methods.

Ex: $$J_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{ix\cos\theta}\, d\theta.$$

$e^{ix\cos\theta}$ is not periodic on $[0,\pi]$, but it is on $[0,2\pi]$.

It can be shown that

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{ix\cos\theta}\, d\theta = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\, d\theta}.$$

See mybessel.m for computing this via the trapezoidal rule.

# Clenshaw - Curtis Quadrature

Special case of Newton-Cotes Formula:

- Interpolate $f$ at Chebyshev nodes
- Integrate each Chebyshev polynomial:

**Details** Approximate $\int_{-1}^{1} f(x)\,dx$ by

$$\approx \int_{-1}^{1} p_n(x)\,dx$$ where $p_n$ is the $\deg n$ interpolant expressed in Chebyshev polynomials:

$$f(x) \approx p_n(x)$$
$$= \sum_{j=0}^{n} c_j T_j(x)$$

and then

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{j=0}^{n} c_j \int_{-1}^{1} T_j(x)\,dx$$

$$= \sum_{j=0}^{n} c_j \int_{-1}^{1} \cos(j \cdot \operatorname{acos} x)\,dx$$

Let $\operatorname{acos} x = \theta$
$dx = -\sin\theta$

$$= \sum_{j=0}^{n} c_j \int_{0}^{\pi} \cos(j \cdot \theta) \sin\theta \, d\theta$$

$$= \sum_{j=0}^{n} c_j \frac{1+\cos(j\pi)}{1-j^2}$$

$$= \sum_{j=0}^{n/2} c_{2j} \frac{2}{1-(2j)^2}$$

Equivalently, we can think about the analogous calculation:

Let $x = \cos\theta$,

then $f(x) = \sum c_k T_k(x)$

$$\Rightarrow f(\cos\theta) = \sum c_k \cos(k \underbrace{\operatorname{acos} x}_{=\theta})$$

$$= \sum c_k \cos(k\theta)$$ ← this is a "cosine" series for the function $f(\cos\theta)$.

And then: $$\int_{-1}^{1} f(x)\,dx \approx \int_0^{\pi} f(\cos\theta)\sin\theta\,d\theta$$

$$\approx \int_0^{\pi} \left(\sum c_k \cos k\theta\right)\sin\theta\,d\theta$$

= same answer as before.

We will return to a "fast" method for computing the $c_k$'s when we talk about the Discrete Fourier Transform, but for now, you can imagine computing them with the composite trapezoidal rule since $f(\cos\theta)\sin\theta$ is a periodic function.