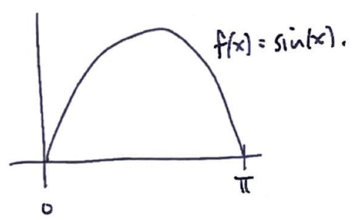
Honors Numerical Analysis

Lecture 19

Approximation in L^2 using orthogonal polynomials

Example: Compute the best quadratic 2-norm approximation to $f(x) = \sin x$ on $(0, \pi)$ with weight function $w(x) = 1$.



The first 3 Legendre polynomials on $(-1,1)$ are:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = x^2 - \frac{1}{3}$$

To define these functions on $(0, \pi)$, let $t = \frac{x+1}{2}\pi$

$$\Rightarrow x = \frac{2t}{\pi} - 1$$

So the shifted orthogonal polynomials are:

$$\tilde{P}_0(t) = 1$$
$$\tilde{P}_1(t) = x = \frac{2}{\pi}t - 1 = \frac{2}{\pi}\left(t - \frac{\pi}{2}\right)$$
$$\tilde{P}_2(t) = x^2 - \frac{1}{3} = \left(\frac{2t}{\pi} - 1\right)^2 - \frac{1}{3}$$
$$= \frac{4t^2}{\pi^2} - \frac{4t}{\pi} + 1 - \frac{1}{3}$$
$$= \frac{4}{\pi}\left(\frac{t^2}{\pi} - t + \frac{\pi}{6}\right).$$

The best approximation is given by the projection of f onto $\operatorname{span}\{\tilde{P}_0, \tilde{P}_1, \tilde{P}_2\}$. Let φ = best 2^{nd}-degree pol. approximant.

$$\Rightarrow \varphi = \frac{(f, \tilde{P}_0)}{(\tilde{P}_0, \tilde{P}_0)}\tilde{P}_0 + \frac{(f, \tilde{P}_1)}{(\tilde{P}_1, \tilde{P}_1)}\tilde{P}_1 + \frac{(f, \tilde{P}_2)}{(\tilde{P}_2, \tilde{P}_2)}\tilde{P}_2.$$

So $(f, \tilde{P}_k)$ can be computed using:

$$\int_0^\pi x \sin x \, dx = \sin x - x\cos x.$$
$$\int_0^\pi x^2 \sin x \, dx = 2x\sin x - (x^2 - 2)\cos x.$$

A bit about orthogonal polynomials

All orthogonal polynomials are eigenfunctions of a differential operator:

$$\underbrace{\mathcal{L}u = -\frac{1}{w}(pu')' + qu}_{\text{Sturm-Liouville Operator}}, \quad \text{with } p, q, w \text{ functions}$$

It can be shown that $\mathcal{L}$ is a self-adjoint linear operator (given suitable boundary conditions), and therefore it has real eigenvalues:

show by $(\mathcal{L}u, v) = (u, \mathcal{L}v)$ + boundary conditions

$$\mathcal{L}u = \lambda u, \quad \lambda \text{ is real.}$$

$\Rightarrow$ The eigenfunctions are orthogonal with respect to the inner product $(u, v) = \int u(x)\, v(x)\, w(x)\, dx$.

Ex: Legendre polynomials

P_n satisfies the Sturm-Liouville eigenfunction problem:

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\right)P_n(x) = -n(n+1)\,P_n(x).$$

Numerical Evaluation of Orth. Pols

All orthogonal polynomials ~~are~~ satisfy a "three-term-recurrence" formula of the form:

$$\varphi_n(x) = (x + a_n)\varphi_{n-1}(x) + b_n \varphi_{n-2} \qquad \text{for } n = 2, 3, \ldots$$

$$\varphi_0(x) = 1$$

$$\varphi_1(x) = x + a_1$$

Alternative form:

$$\varphi_{n+1} = a_n x \varphi_n + b_n \varphi_{n-1}$$

Since they are orthogonal, the coefficients are given by

$$a_n = -\frac{(x P_{n-1}, P_{n-1})}{(P_{n-1}, P_{n-1})}$$

$$b_n = -\frac{(P_{n-1}, P_{n-1})}{(P_{n-2}, P_{n-2})}$$

<u>Ex</u>: Legendre: $P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$

Chebyshev: $T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$

Hermite: $H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$

Stability comes from an analysis of $\begin{pmatrix} 0 & 1 \\ b_n & a_n x \end{pmatrix} \begin{pmatrix} \varphi_{n-1} \\ \varphi_n \end{pmatrix} = \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}$

Numerical Integration

Almost no integral can be computed analytically, they must be evaluated numerically.

Ex:

$$\underbrace{\operatorname{erf}(x)}_{\text{error function}} = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

Application ODEs, initial value problems

$$(*) \quad \begin{aligned} y'(t) &= f(t) \\ y(0) &= y_0 \end{aligned}$$

The analytic solution is

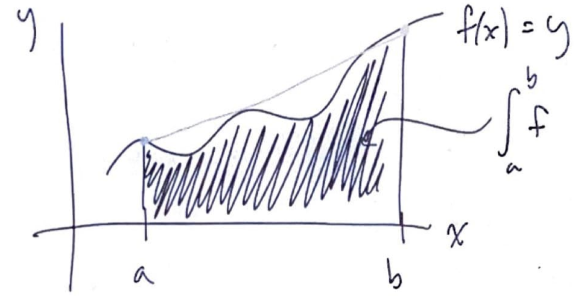
$$y(t) = y_0 + \int_0^t f(\tau)\,d\tau$$

All numerical methods for solving $(*)$ are based on numerically approximating the integral.

More to come later...

The Trapezoidal (Trapezium) Rule

The most basic numerical integration technique:



Option 1: Approximate f on $[a,b]$ by a line, integrate the line.

Area under this line is $\underbrace{(b-a)}_{\text{width}} \underbrace{\frac{(f(a)+f(b))}{2}}_{\text{average of edge heights.}} \approx \int_a^b f(x)\,dx.$

Area of a trapezoid.

① f was approximated by a line which interpolated f at $x=a$ and $x=b$.

② This interpolating line was integrated.

The trapezoidal rule is a special case of more general numerical integration formulas known as Newton-Cotes rules:

<u>Idea</u> Interpolate f on $[a,b]$, and then integrate the interpolating polynomial.

This can be done with arbitrarily high degree (high order interpolation) but remember: the interpolating polynomial may suffer from <u>Runge's Phenomena</u>.