Approximation in \( L^2 \) using orthogonal polynomials

Example: Compute the best quadratic 2-norm approximation to \( f(x) = \sin x \) on \((0, \pi)\) with weight function \( w(x) = 1 \).

\[
\int_0^\pi \sin x \, dx = \sin x - x \cos x.
\]

The first 3 Legendre polynomials on 
\((-1, 1)\) are:
\[
P_0(x) = 1
\]
\[
P_1(x) = x
\]
\[
P_2(x) = x^2 - \frac{1}{3}
\]

To define these functions on \((0, \pi)\), let \( t = \frac{x+1}{2} \)
\[\Rightarrow x = 2t - 1\]

So the shifted orthogonal polynomials are:
\[
\tilde{P}_0(t) = 1
\]
\[
\tilde{P}_1(t) = t = \frac{1}{2} (2t - 1) = \frac{1}{2} (t - \frac{1}{2})
\]
\[
\tilde{P}_2(t) = t^2 - \frac{1}{3} = (\frac{1}{2} (2t - 1))^2 - \frac{1}{3}
\]
\[= \frac{4t^2}{4} - \frac{4t}{4} + \frac{1}{4} - \frac{1}{3}
\]
\[= \frac{4t}{4} (\frac{1}{2} (2t - 1) + \frac{1}{4})
\]

The best approximation is given by the projection of \( f \) onto \( \text{span} \{ \tilde{P}_0, \tilde{P}_1, \tilde{P}_2 \} \). Let \( q = \) best 2nd-degree pol. approximation
\[\Rightarrow q = \frac{\langle f, \tilde{P}_0 \rangle \tilde{P}_0 + \langle f, \tilde{P}_1 \rangle \tilde{P}_1 + \langle f, \tilde{P}_2 \rangle \tilde{P}_2}{\langle \tilde{P}_0, \tilde{P}_0 \rangle} \frac{\tilde{P}_0}{\langle \tilde{P}_0, \tilde{P}_0 \rangle} + \frac{\langle f, \tilde{P}_1 \rangle \tilde{P}_1}{\langle \tilde{P}_1, \tilde{P}_1 \rangle} + \frac{\langle f, \tilde{P}_2 \rangle \tilde{P}_2}{\langle \tilde{P}_2, \tilde{P}_2 \rangle}
\]

So \( \langle f, \tilde{P}_0 \rangle \) can be computed using:
\[\int_0^\pi x \sin x \, dx = \sin x - x \cos x.
\]
\[\int_0^\pi x^2 \sin x \, dx = 2x \sin x - (x^2 - 2) \cos x.
\]
A bit about orthogonal polynomials

All orthogonal polynomials are eigenfunctions of a differential operator:

\[ L u = \frac{1}{w} (p u')' + q u, \quad \text{with } p, q \text{ function} \]

Storm-Liouville Operator

It can be shown that \( L \) is a self-adjoint linear operator (given suitable boundary conditions), and therefore it has real eigenvalues:

\[ L u = \lambda u, \quad \lambda \text{ is real} \]

\[ \Rightarrow \text{The eigenfunctions are orthogonal with respect to the inner product } (u, v) = \int u(x) v(x) w(x) \, dx. \]

Ex: Legendre polynomials

\( P_n \) satisfies the Storm-Liouville eigenfunction problem:

\[ \frac{d}{dx} \left( (1-x^2) \frac{d}{dx} \right) P_n(x) = -n(n+1) P_n(x). \]
Numerical Evaluation of Orth. Polys

All orthogonal polynomials satisfy a "three-term-recurrence" formula of the form:

\[ q_n(x) = (x + a_n)q_{n-1}(x) + b_n q_{n-2} \quad \text{for } n = 2, 3, \ldots \]

\[ q_0(x) = 1 \]

\[ q_1(x) = x + a_1 \]

Since they are orthogonal, the coefficients are given by

\[ a_n = \frac{(x, P_{n-1}, P_{n-1})}{(P_{n-1}, P_{n-1})} \]

\[ b_n = \frac{(P_{n-1}, P_{n-1})}{(P_{n-2}, P_{n-2})} \]

**Ex:** Legendre: \( P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x) \)

Chebyshev: \( T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) \)

Hermite: \( H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) \)

Stability comes from an analysis of

\[
\begin{pmatrix}
0 & 1 \\
b_n & a_n \\
\end{pmatrix}
\begin{pmatrix}
q_n(x) \\
q_{n+1}(x) \\
\end{pmatrix}
= 
\begin{pmatrix}
q_n \\
q_{n+1} \\
\end{pmatrix}
\]


Numerical Integration

Almost no integral can be computed analytically, they must be evaluated numerically.

Ex:
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt
\]
error function

Application ODEs, initial value problems

\[(x) \quad y'(t) = f(t) \quad \text{The analytic solution is} \quad y(t) = y_0 + \int_0^t f(\tau) d\tau \]

All numerical methods for solving \((x)\) are based on numerically approximating the integral.

More to come later...
The Trapezoidal (Trapezium) Rule

The most basic numerical integration technique:

\[ f(x) = y \]

\[ \int_a^b f(x) \, dx \]

Area under this line is

\[ \frac{(b-a)(f(a) + f(b))}{2} \]

Option 1: Approximate \( f \) on \([a, b]\) by a line, integrate the line.

Area of a trapezoid.

1. \( f \) was approximated by a line,
   which interpolated \( f \) at \( x=a \) and \( x=b \).

2. This interpolating line was integrated.

The trapezoidal rule is a special case of more general numerical integration formulas known as Newton-Cotes rules: 

\[ \text{Idea: Interpolate } f \text{ on } [a, b], \text{ and then integrate the interpolating polynomial.} \]

This can be done with arbitrarily high degree (high order interpolation) but remember: the interpolating polynomial may suffer from Runge's Phenomena.