Function approximation in the 2-norm.

Goal For a given function f on $[a,b]$, find $p_n \in P_n$ that $\|f - p_n\|_2$ is as small as possible.

$\Rightarrow$ We previously derived that this is a least squares problem and therefore there exists a constructive solution, i.e.

If $p_n(x) = c_0 \varphi_0(x) + c_1 \varphi_1(x) + \ldots + c_n \varphi_n(x)$

form a basis for P_n, then the coefficients c_j are obtained by solving a linear system.

Finally, if the functions φ_j are orthonormal, then the coefficients c_j are merely inner products:

$$c_j = (f, \varphi_j).$$ (analogous to Gram-Schmidt, or orthogonal projections).

This approximation of f is equivalent to finding its orthogonal projection onto P_n under the inner product:

$$(f,g) = \int_a^b f(x)\, g(x)\, w(x)\, dx.$$

Definition If the sequence of polynomials $\varphi_0, \varphi_1, \ldots, \varphi_n$, with $\deg \varphi_j = j$, on the interval (a,b) satisfies

$$\int_a^b \varphi_j(x)\, \varphi_k(x) = 0 \quad \text{if } j \neq k,$$ then $\varphi_0, \ldots, \varphi_n$ is a system of orthogonal polynomials (with weight $w(x) = 1$).

(Likewise we would define $(f,g) = \int_a^b f(x)\, g(x)\, w(x)\, dx$.)

Example: Find q_0, q_1, q_2 on $[-1,1]$ with weight function $w(x)=1$.

Set $q_0(x)=1$.

$q_1(x) = ax+b$.

If $(q_0, q_1)=0$ then $\int_{-1}^{1} 1\cdot(ax+b)\,dx = 0$

$2b=0 \Rightarrow b=0$.

Set $q_1(x)=x$.

Let $q_2(x) = x^2+bx+c$

Two conditions must be satisfied:

$$\int_{-1}^{1} q_0(x)q_2(x)\,dx=0 \qquad\qquad \int_{-1}^{1} q_1(x)q_2(x)\,dx=0$$

$$\int_{-1}^{1}(x^2+bx+c)\,dx=0 \qquad\qquad \int_{-1}^{1} x(x^2+bx+c)\,dx=0$$

$$\frac{2}{3}+2c=0 \qquad\qquad \frac{2}{3}b=0 \Rightarrow b=0.$$

$$c=\frac{-1}{3}$$

So we set $q_2(x) = x^2-\frac{1}{3}$.

So by construction, 1, x, and $x^2-\frac{1}{3}$ are orthogonal on $[-1,1]$.
We could do the same calculations for $q_3, q_4, \ldots$
The resulting polynomials are known as Legendre Polynomials. They form an orthogonal basis for all of $\underbrace{L^2[-1,1]}$ under the inner product $(f,g) = \int_{-1}^{1} f(x)g(x)\,dx$.

$f\in L^2[-1,1]$ iff $\underbrace{\int_{-1}^{1}(f(x))^2\,dx < \infty}_{\|f\|_2^2 < \infty}$.

Legendre polynomials can also be constructed another way as well: the Gram-Schmidt process.

Start with $P_0(x) = 1$, $P_1(x) = x$.] automatically orthogonal.

Set $m_2(x) = x^2$. ← linearly independent from P_0, P_1.

For Gram-Schmidt: $P_2 = m_2 - \mathrm{Proj}_{\{P_0, P_1\}} m_2$

$$= m_2 - \frac{(m_2, P_0)}{(P_0, P_0)} P_0 - \frac{(m_2, P_1)\, P_1}{(P_1, P_1)}$$

So $P_2(x) = x^2 - \frac{2}{3}\frac{1}{2} \cdot 1 - 0$

$= x^2 - \frac{1}{3}$ Exactly the same as before.

So in general, compute

$$P_n = m_n - \sum_{\ell=0}^{n-1} \frac{(m_n, P_\ell)}{(P_\ell, P_\ell)} P_\ell$$

$$P_n(x) = x^n - \sum_{\ell=0}^{n-1} \frac{P_\ell(x)}{\|P_\ell\|_2^2} \int_{-1}^{-1} x^n P_\ell(x)\, dx$$

These polynomials can be scaled to any interval.

If $\int_{-1}^{1} P_j(x) P_k(x)\, dx = 0$ if $j \neq k$, then it is easy

to show that $\int_a^b P_j(t) P_k(t)\, dt = 0$ if $j \neq k$

where $t = \left(\frac{x+1}{2}\right)(b-a) + a$.

Ex: Chebyshev Polynomials

We know that $\int_0^{\pi} \cos mt \cos nt\, dt = 0$ if $m \neq n$.

Let $t = \arccos x$, $dt = \frac{1}{\sqrt{1-x^2}} dx$, $t: -1 \to 1$

$$\Rightarrow \int_{-1}^{1} \underbrace{\cos(m \operatorname{acos} x)}_{T_m(x)} \cdot \underbrace{\cos(n \operatorname{acos} x)}_{T_n(x)} \cdot \frac{dx}{\sqrt{1-x^2}} = 0 \quad \text{if } m \neq n.$$

$$\Rightarrow \int_{-1}^{1} T_m(x)\, T_n(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0 \quad \text{if } m \neq n.$$

Therefore, the functions $T_0, T_1, T_2, \ldots$ are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$.

<u>Theorem</u> If $\int_a^b |f(x)|^2 w(x)\, dx < \infty$ (ie. $f \in L_w^2[a,b]$), there is a <u>unique</u> degree n polynomial p_n such that

$$\| f - p_n \|_{w,2} = \min_{q \in P_n} \| f - q \|_{w,2}$$

where $\| f \|_{w,2}^2 = \int_a^b |f(x)|^2 w(x)\, dx.$

<u>Proof</u>: Gram-Schmidt, solve directly for the coefficients of the associated orthogonal polynomial expansion to compute the approximation.

Note: This is just linear algebra!

Famous sets of orthogonal polynomials:

$w(x)$	(a,b)	Polynomial
1	$(-1,1)$	Legendre
$1/\sqrt{1-x^2}$	$(-1,1)$	Chebyshev
e^{-x}	$(0,\infty)$	Laguerre
e^{-x^2}	$(-\infty,\infty)$	Hermite.