Function approximation in the $L^2$-norm.

Goal: For a given function $f$ on $[a, b]$, find $p_n \in P_n$ such that $\|f - p_n\|_2$ is as small as possible.

We previously derived that this is a least squares problem and therefore there exists a constructive solution, i.e.,

If $p_n = c_0 q_0(x) + c_1 q_1(x) + \ldots + c_n q_n(x)$

form a basis for $P_n$, then the coefficients $c_j$ are obtained by solving a linear system. Finally, if the functions $q_j$ are orthonormal, then the coefficients $c_j$ are merely inner products:

$$c_j = (f, q_j).$$ (analogous to Gram-Schmidt, or orthogonal projections).

This approximation of $f$ is equivalent to finding its orthogonal projection onto $P_n$ under the inner product:

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)\,dx.$$

Definition: If the sequence of polynomials $q_0, q_1, \ldots, q_n$, with $\deg q_j = j$, on the interval $[a, b]$ satisfies

$$\int_a^b q_j(x)q_k(x)\,w(x)\,dx = 0 \quad \text{if} \quad j \neq k,$$

then $q_0, \ldots, q_n$ is a system of orthogonal polynomials (with weight $w(x) = 1$).

(Likewise we could define $(f, g) = \int_a^b f(x)g(x)w(x)\,dx$.)
Example: Find $q_0, q_1, q_2$ on $[-1, 1]$ with weight function $w(x) = 1$. Set $q_0(x) = 1$.

$q_1(x) = ax + b$. 

If $(q_0, q_1) = 0$ then

$$
\int_{-1}^{1} (ax + b) \, dx = 0 \\
2b = 0 \implies b = 0.
$$

Set $q_1(x) = x$.

Let $q_2(x) = x^2 + bx + c$

Two conditions must be satisfied:

$$
\int_{-1}^{1} q_0(x) q_2(x) \, dx = 0 \\
\int_{-1}^{1} (x^2 + bx + c) \, dx = 0
$$

$$
\frac{2}{3} + 2c = 0 \\
c = -\frac{1}{3}
$$

So we set $q_2(x) = x^2 - \frac{1}{3}$.

So by construction, $1, x, x^2$ are orthogonal on $[-1, 1]$. We could do the same calculation for $q_3, q_4, ...$

The resulting polynomials are known as Legendre Polynomials. They form an orthogonal basis for all of $L^2[-1, 1]$ under the inner product $(f, g) = \int_{-1}^{1} f(x) g(x) \, dx$.

$f \in L^2[-1, 1]$ iff

$$
\int_{-1}^{1} (f(x))^2 \, dx < \infty. \\
\norm{f}_{L^2} < \infty
$$
Legendre polynomials can also be constructed another way as well: the Gram-Schmidt process.

Start with \( P_0(x) = 1 \), \( P_1(x) = x \). Automatically orthogonal.

Set \( m_2(x) = x^2 \). Linearly independent from \( P_0, P_1 \).

For Gram-Schmidt: \( P_2 = m_2 - \text{Proj}_{P_0, P_1}m_2 \)

\[
= m_2 - \frac{(m_2, P_0)}{(P_0, P_0)} P_0 - \frac{(m_2, P_1)}{(P_1, P_1)} P_1
\]

So \( P_2(x) = x^2 - \frac{2}{3} \cdot \frac{1}{2} \cdot 1 - 0 \)

\[= x^2 - \frac{1}{3} \]  Exactly the same as before.

So in general, compute

\[
P_n = m_n - \sum_{l=0}^{n-1} \frac{(m_n, P_l)}{(P_l, P_l)} P_l
\]

\[
P_n(x) = x^n - \sum_{l=0}^{n-1} \frac{P_l(x)}{lP_l !^2} \int_0^1 x^n P_l(x) \, dx
\]

These polynomials can be scaled to any interval.

If \( \int_0^1 P_j(x) P_k(x) \, dx = 0 \) if \( j \neq k \), then it is easy
to show that \( \int_0^1 P_j(t) P_k(t) \, dt = 0 \) if \( j \neq k \)

where \( t = \left( \frac{x+1}{2} \right)(b-a) + a \).

Ex: Chebyshev Polynomials

We know that \( \int_0^\pi \cos mx \cos nt \, dt = 0 \) if \( m \neq n \).

Let \( t = \cos x \), \( dt = -\sin x \, dx \), \( t: -1 \to 1 \).
\[ \int_{-1}^{1} \frac{\cos(m \cos x) \cdot \cos(n \cos x)}{\sqrt{1-x^2}} \, dx = 0 \quad \text{if } m \neq n. \]

\[ \Rightarrow \int_{-1}^{1} T_m(x) T_n(x) \frac{1}{\sqrt{1-x^2}} \, dx = 0 \quad \text{if } m \neq n. \]

Therefore, the functions \( T_0, T_1, T_2, \ldots \) are orthogonal on \([-1, 1]\) with respect to the weight function \( w(x) = \frac{1}{\sqrt{1-x^2}} \).

**Theorem** If \( \int_{-1}^{1} |f(x)|^2 \, w(x) \, dx < \infty \) (i.e., \( f \in L^2([-1, 1]) \)), there is a unique degree \( n \) polynomial \( p_n \) such that

\[ \| f - p_n \|_{L^2} = \min_{q \in P_n} \| f - q \|_{L^2} \]

where \( \| f \|_{L^2} = \left( \int_{-1}^{1} |f(x)|^2 \, w(x) \, dx \right)^{1/2} \).

**Proof:** Gram-Schmidt, solve directly for the coefficients of the associated orthogonal polynomial expansion to compute the approximation.

**Note:** This is just linear algebra!

**Famous sets of orthogonal polynomials:**

<table>
<thead>
<tr>
<th>( w(x) )</th>
<th>( [a, b] )</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>([-1, 1])</td>
<td>Legendre</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{1-x^2}} )</td>
<td>([-1, 1])</td>
<td>Chebyshev</td>
</tr>
<tr>
<td>( e^x )</td>
<td>([0, \infty))</td>
<td>Laguerre</td>
</tr>
<tr>
<td>( e^{-x^2} )</td>
<td>((-\infty, \infty))</td>
<td>Hermite</td>
</tr>
</tbody>
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